AN ANALYSIS ON THE EFFECTS OF A CHANGE OF WAGES ON CAPITAL AND CONSUMPTION

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ABSTRACT.—In this paper we analize the dynamics associated with an intertemporal choice model, assuming that the wages obtained outside the capital market are determined exogenously. They are supposed to be the parameter of the model. Choosing an S-shaped interest function and suitable values of the preference rate, the dynamical system associated with the model has a non-hyperbolic fixed point. The aim of the study is to determine whether it is possible, from the original equilibrium point, to attain the steady state of the new dynamical system obtained after a change on the value of the wages. The problem is analyzed by means of the center manifolds associated with the non-hyperbolic fixed points. We consider both permanent and temporary shifts on the value of wages.

1. INTRODUCTION

In the economic dynamic's literature it is quite frequent to meet models involving some parameters. It is possible that the behavior of the dynamical system would be modified by a shift on the value of the parameter. Examples of this situation can be founded in the vast liberature about chaos (see, for example, B. Mizrach (1992)).

However, there are other kind of models in which a change on a parameter does not modify the behavior of the trajectories in the phase plane substantially. In that case we can abbord the following question. If the dynamical system has a steady state, endpoint of a stable optimal path, and the value of the parameter changes, the system will present a new fixed point. The problem is whether the dynamical system will be able to reach the new stable optimal path. In case of a temporary change in the value the objective will be to attain the original equilibrium. This situation must be analized in the context of the optimal control theory.

Similar problems are treated in J. Pitchford (1989) and in S. Turnovsky and P. Sen (1989). Both papers examine an akin question to the

Harberger-Laursen-Metzler effect which has been questioned by several authors. This effect asserts that a permanent deterioration in the terms of trade will worsen the current account. Working with different models, both authors base their conclusions, opposite to Harberger-Laursen-Metzler effect in some cases, on the existence of a unique saddle point in the dynamical system obtained through the necessary conditions of the Maximum Principle.

Under this assumption, if a parameter changes temporarily or permanently it will be always possible to reach the optimal path again, though the way of made it can be quite different. The possibility of carrying out the adjustement in different ways is the origin of the controversy on the Harberger-Laursen-Metzler effect in this kind of models.

In this paper we have chosen a continuous model of intertemporal choice autonomous and with infinite horizon. The budget constraint includes the wages of a consumer, that we assume exogenously determined and it will be the parameter of the model[1]. We consider that the dynamical system arising from necessary conditions has a unique fixed point, which is neither a saddle point nor an unstable equilibrium. Now the center manifold associated to the stationary solution will play a similar role to the stable manifold when the dynamical system admits a saddle point.

The paper is organized as follows. Section 2 discusses general features of the model, while section 3 looks at responses to permanent changes in wages. Next section examines temporary wages shifts. The conclusions are discussed in the final section 5.

2. THE MODEL AND FIRST RESULTS

We consider an individual that seeks the consumption rate for each time that will maximize his discounted utility stream over an infinite interval of time. The utility of consumption $u[c(t)]$, at each moment t, is an increasing concave function: $u(c) \in C^3$, $u'(c) > 0$, $u''(c) < 0$, for all c positive. Following I. Fisher we assume that the consumer is impatient, then the future utility is discounted at a rate of preference r. The objective is:

$$\max_{c} \int_0^\infty u[c(t)]e^{-rt}dt,$$

subject to a cash flow constraint. The individual derives current income from exogenously determined wages, a, and from interest earnings, $i(k)$, on his holdings of capital assets $k(t)$. The wages are treated as a parameter

1 This model appears in M. Kamien and N. Schwarts (1981), p. 25 and also in E. Silberberg (1990), p. 632, with finite horizon and without wages in the budget constraint.

which is subject to current and future known fluctuations. The interest function $i(k)$ satisfies $i(0) = 0$ and $i'(k) > 0$, for k positive. We assume that $i(k)$ is non-linear opposite to the classical assumption of the intertemporal choice. Now, income from interest and wages are allocated to consumption and investment:

$$i(k) + a = c + \dot{k}, \qquad k(0) = 0.$$

We assume that the consumption and capital are positive for all moments of time.

This problem can be solved by means of the Maximum Principle of Pontryagin. The hamiltonian of the model is:

$$H = u(c) + \psi[i(k) - c + a]$$

where $\psi(t)$ is the costate variable associated with the state variable $k(t)$.

Now, assuming a consumption of class C^3, the necessary conditions yield the following system of differential equations:

$$\dot{c} = \frac{u'(c)}{u''(c)}(r - i'(k)),$$

$$\dot{k} = i(k) - c + a. \tag{1}$$

Let us suppose that the system above presents a unique steady state (k^*, c^*). If the wages a stay constant, (k^*, c^*) will satisfy: $r = i'(k^*)$, $c^* = i(k^*) + a$. From these relations we can assert that the stationary value of capital k^* is not modified when parameter a changes, but consumption takes a new value according to the shift in a. The behavior of the dynamical system can be derived by means of the analysis of the associated phase plane. Then, it will be necessary to study the stability of the fixed point. The Jacobian matrix evaluated in the stationary state is:

$$J = \begin{pmatrix} 0 & -\dfrac{u'(c^*)}{u''(c^*)} i''(k^*) \\ -1 & r \end{pmatrix}.$$

Thus, we have three different cases. First, if $i''(k^*)$ is positive, the fixed state is a saddle point. Now the necessary conditions of optimality are also sufficient. The optimal solution will be the stable manifold of the fixed point. Secondly, if $i''(k^*)$ is negative, the fixed point is unstable. Finally, if $i''(k^*)$ is equal to zero, then one of the eigenvalue of the Jacobian matrix is zero and the other one equals the preference rate. This last case can occur if, for example, the interest function is S-shaped and $r = \max_k i'(k)$.

Let we remark that independently of the value of $i''(k)$ at k^* the phase plane does not contain closed orbits, since the Jacobian matrix has traze not null.

On the rest of the paper we study the case $i''(k^*) = 0$. In order to determine the phase diagram, we will take into account the Center Manifold Theorem for flows[2]. It asserts, that there exist a C^3-class unstable invariant manifold, W^u, tangent to E^u at (k^*, c^*) and a C^2-class center manifold, W^c, tangent to E^c at (k^*, c^*). Here E^u and E^c denote the unstable and center eigenspaces, respectively.

Manipulating the linearized system we obtain the following expressions for the invariant subspaces:

$$E^u(k^*, c^*) = \{(k, c): c - c^* = 0\},$$

$$E^c(k^*, c^*) = \{(k, c): -(c - c^*) + r(k - k^*) = 0\}.$$

The direction of the flow in E^u is completely determined, but we cannot assign a direction of the flow in $E^c(W^c)$ without specific information on the higher-order terms of the dynamical system near (k^*, c^*). This information is obtained by expanding the right hand side of (1) by means of a Taylor series. We get the following second-order approximation of the dynamical system at (k^*, c^*):

$$\dot{c} = \frac{-1}{2} i'''(k^*)(k - k^*)^2 \phi(c^*),$$

$$\dot{k} = (c^* - c) + r(k - k^*), \tag{2}$$

where $\phi(c) = \dfrac{u'(c)}{u''(c)} < 0.$

In order to derive a formulation of (2) easier to handle, first we translate the origin to the critical point. Afterwards we make the change of variables,

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix},$$

where $p = c - c^*$, $q = k - k^*$ and the matrix (2 x 2) has as columns the eigenvectors associated with eigenvalues 0 and r, respectively. Then, we arrive at this new formulation of the dynamical system:

2 Guckenheimer and Holmes (1983), p. 127.

$$\dot{n} = \frac{-1}{2r} i'''(k^*)\phi(c^*)(n+m)^2,$$

$$\dot{m} = rm + \frac{1}{2} i'''(k^*)\phi(c^*)(n+m)^2.$$

Following Guckenheimer and Holmes[3], the approximation to the center manifold can be stated as:

$$W^c(k^*, c^*) = \left\{ (m, n) \colon m = \frac{-1}{2r^2} \phi(c^*) i'''(k^*) n^2 \right\}.$$

Thus depending on the sign of $i'''(k^*)$ we have two possible representations of W^c. The direction of the flow in W^c is obtained from Henry-Carrs' theorem[4].

These results are already enough to determine the behavior of the dynamical system on the phase plane near (k^*, c^*). The figure (1) shows, with coordinates (m, n), the unstable subspace E^u: $n = 0$[5], and the approximated center manifold W^c (I happens if $i'''(k^*)$ is positive and II if $i'''(k^*)$ is negative).

Let us note that the stable branch of the center manifold should be the optimal solution of the model. In fact, with the help of the phase diagram, one can see that there are some trajectories only admissible in finite horizon and other ones involving a non-optimal behavior of consumption under the assumptions maded on $u[c(t)]$.

3. PERMANENT CHANGE IN WAGES

In this section we assume that the individual at $t = 0$ has wages a_1, but he knows that at a later time $T > 0$ his wages will be $a_2 \neq a_1$. Initially his levels of consumption and capital are on the optimal trajectory, which tends asymptotically to (k_1^*, c_1^*). This last value is the equilibrium point of the dynamical system with $a = a_1$. But when the parameter a takes the value a_2, the consumer wants to reach the new optimal trajectory, that is, the new path tending asymptotically to the equilibrium value (k_2^*, c_2^*), of the dynamical system with $a = a_2$.

3 Gunckenheimer and Holmes (1983), p. 127.

4 Gunckenheimer and Holmes (1983), p. 131.

5 We do not consider the second-order approximation of W^u in order to simplify the graphical representations.

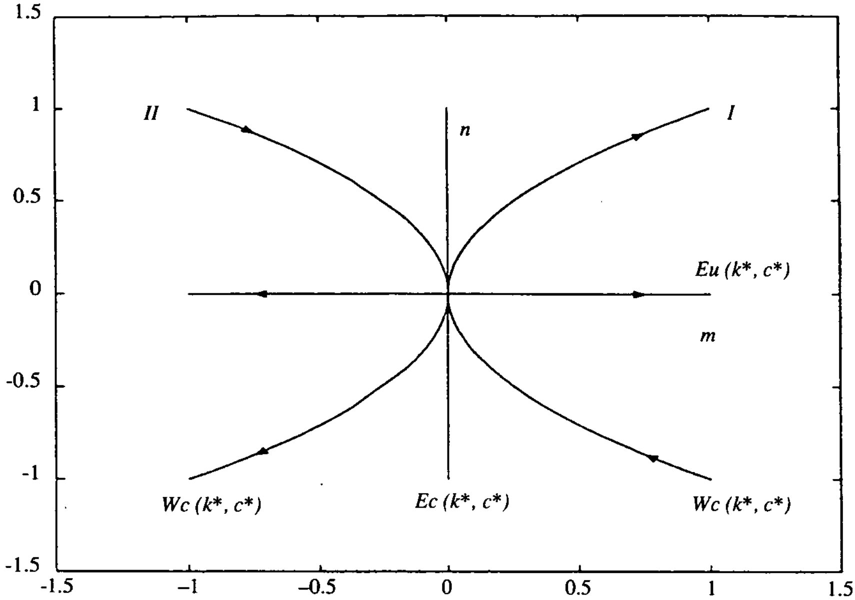

FIG. 1

Thus, with this information, added to the dynamical system in some way and assuming perfect foresight, the consumer attempts to modify the trajectory that he is following, in order to attain the situation which will be optimal afterwards, when the value of the parameter changes. For that reason, the process of adjustment is conditioned by the expectations of the long-term steady state.

The problem can be studied with the help of phase diagrams. Without loss of generality we assume that $a_1 > a_2$, according to the results obtained in the previous section and using the coordinates (m, n) we have that assuming that, (k_1^*, c_1^*) equals to $(0, 0)$, $(k_2^*, c_2^*) = \left(-\frac{1}{r}(a_2 - a_1), \frac{1}{r}(a_2 - a_1)\right)$ where $(c_2^* - c_1^*) = a_2 - a_1$.

From straightforward calculations the expressions of the center manifold and the unstable subspace at (k_2^*, c_2^*) become:

$$E^u(k_2^*, c_2^*) = \left\{(m, n): n = \frac{1}{r}(a_2 - a_1)\right\}.$$

$$W^c(k_2^*, c_2^*) = \left\{(m, n): m + \frac{1}{r}(a_2 - a_1) = \alpha_1^{(2)}\left(n - \frac{1}{r}(a_2 - a_1)\right)^2\right\}.$$

where $\alpha_1^{(2)} = \frac{-1}{2r^2}\phi(c_2^*)i'''(k_2^*)$

The figure 2 shows the superposition of the phase diagrams, before and after the change of wages. We remind that the branch of type *I* happens if $i'''(k_1^*) = i'''(k_2^*) > 0$ and type II in case that $i'''(k_1^*) = i'''(k_2^*) < 0$.

Under the assumption that a_1 is greater than a_2, from figure (2) we can note that it is not possible to reach (k_2^*, c_2^*) from a position near to (k_1^*, c_1^*) if $i'''(k_1^*)$ is positive, since the direction of the movement along the branch of $W^c(k_2^*)$ below $m = 0$ goes away of the fixed point. If $a_2 > a_1$ and $i'''(k_2^*, c_2^*)$ is negative the analogous figure shows that neither is able the consumer to reach the optimal path.

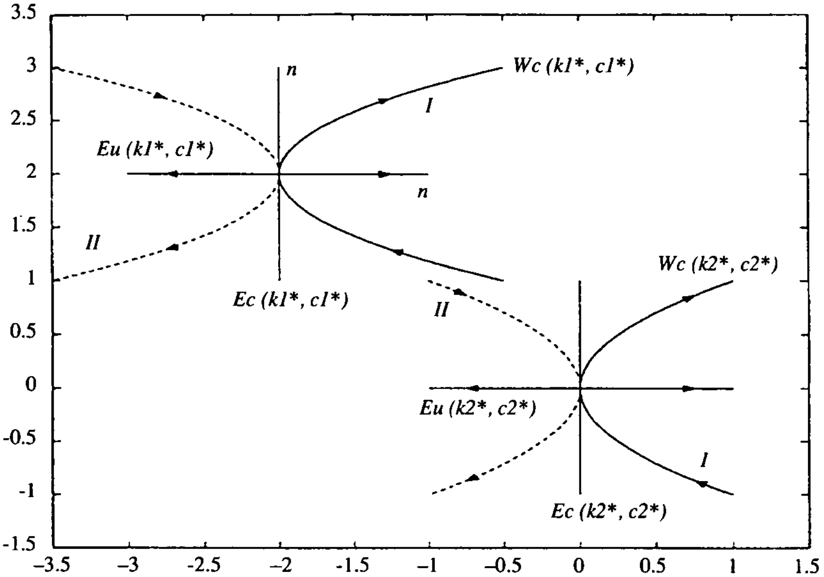

FIG. 2

However, if $i'''(k_1^*)$ is negative, as is the case in a S-shaped interest function when k_1^* is the maximum of the marginal interest function, then (k_2^*, c_2^*) can be reached from (k_1^*, c_1^*). The period of adjustment required by that process consists on modifying the trajectory that the consumer follows when $a = a_1$, bearing in mind the information that he has. Afterwards, the individual will go along the optimal way.

In fact, if at $t = 0$ the consumer has values of consumption and capital next to (k_1^*, c_1^*), then he can follow the flow determined by E^u, that connects with the optimal trajectory associated with $a = a_2$ at the point $(\hat{k}(0), \hat{c}(0))$ (Figure 3).

If the levels of consumption and when the wages change at T are $(\hat{k}(0), \hat{c}(0))$, then the consumer will surely attain the new optimal way. Let us note that he can choose several trajectories whenever he seeks to find a point on $W^c(k_2^*, c_2^*)$ at T such that tends asymptotically towards (k_2^*, c_2^*). The choice depends on the value of T.

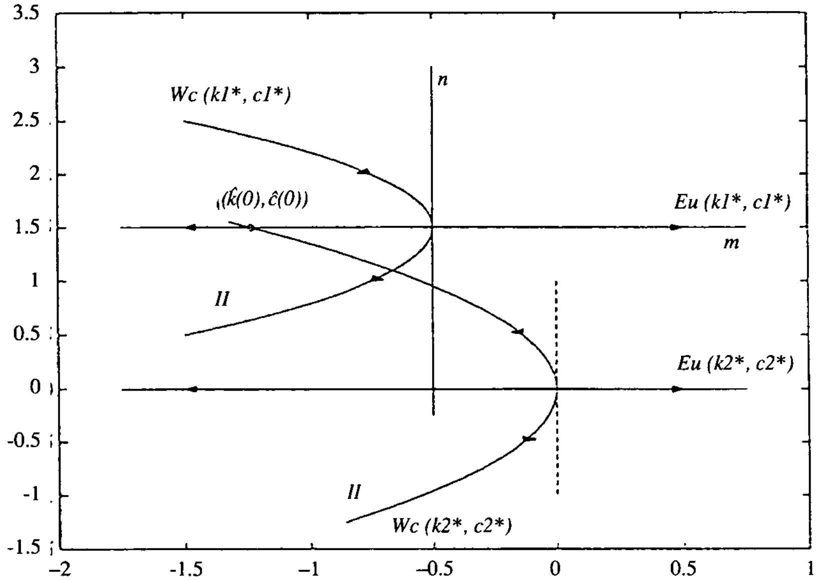

FIG. 3

We assume that the consumer chooses $E^u(k_1^*, c_1^*)$ and transleting the results obtained from variables m and n to variables c and k, we can assert that, until T the consumption stays constant $c = c_1^*$ and as a result, the capital decreases.

Under this assumption, we can also determine the time that the capital takes to reach $(\hat{k}(0), \hat{c}(0))$ from a position near (k_1^*, c_1^*). The point $(\hat{k}(0), \hat{c}(0))$ must satisfy the following conditions:

$$\hat{k}(0) = c_1^*, \quad \hat{k}(0) > k_1^*$$

that is, it is on $E^u(k_1^*, c_1^*)$ and, moreover, it will be on $W^C(k_2^*, c_2^*)$:

$$\hat{k}(0) = -\frac{1}{r}(a_2 - a_1)\left[1 - \alpha_1^{(2)} \frac{1}{r}(a_2 - a_1)\right]$$

In order to determine the time that the capital spends on the previous process we need to find a solution of the dynamical system such that satisfies the boundary conditions:

$$(k(0), c(0)) \in E^u(k_1^*, c_1^*),$$

$$(k(T), c(T) = (\hat{k}(0), \hat{c}(0)),$$

where $k(0)$, $c(0)$ are the initial conditions. If we choose $k(0) = k_1^* - \mu$ with $\mu > 0$ and $c(0) = c_1^*$ and since $c = c_1^*$ in the interval $[0, T)$, the problem lies in solving the differential equation $\dot{k} = i(k) - c_1^* + a_1$ and using the boundary conditions to determine the constant of integration and the unknown time T.

In the long-term analysis the consumption is always decreasing, it falls from $c = c_1^*$ to $c = c_2^*$, and its variation rate is not affected by the value of wages. The capital is still decreasing because the consumption is still too high. In the long-term the marginal rate of interest is less than the rate of preference, and the variation rate of the capital is proportional to the value of wages.

The capital does not always decrease in long-term, since it reaches a minimum which satisfies $i(k) + a_2 = c$ and the behavior then goes back to attain the old value of capital $k_2^* = k_1^*$.

Summing up, we can assert that, in first instance, consumption stays constant and the capital falls. This falling-off of capital induces a new drop on consumption, but the capital does not grow until the consumption is small enough.

4. TEMPORAL CHANGE IN WAGES

Next we study the effects of a temporal instead of a permanent change in wages. We assume that since the consumer knows that the change is only temporary, his objective must be different from the previons one. Now, first the dynamical system evolves during an initial period of time with the wages value a_1. Next the parameter changes taking a value a_2 and after a second period of time the wages return again and forever to a_1. The consumer knows exactly the length of all periods involved in the process. Let us note that the dynamical system associated to the model is different in the central period from the first and third ones.

In order to simplify the analysis we start at the moment of the first shift on the wages value, keeping in mind that the attainable positions must be reached by means of the flow of the original dynamical system. Mathematically this can be done through a simple time reparametrization.

So, at $t = 0$ the wages have the value a_1 and the steady state of the dynamical system associated is (k_1^*, c_1^*). During the time interval $(0, T)$, when parameter a takes the value a_2, the equilibrium point will be (k_2^*, c_2^*) and when $t \geq T$, the stationary situation will be (k_1^*, c_1^*), again. Now the phase plane will be different in $(0, T)$ from $[T, \infty)$ and $t = 0$.

We consider two different situations that yields the same result. The consumer can be or not in the optimal trayectory at $t = 0$ but he knows the future with security. In the first case, he must be able to reach a position in the phase plane of the dynamical system with $a = a_1$ such that, when T elapses, and the system evolves with $a = a_2$, he finds the stable branch of $W^c(k_1^*, c_1^*)$ again. If the individual initially is not on this stable branch he must look for analogous conditions on the phase diagram.

Without loss of generality now we assume that $a_2 > a_1$[6]. In the opposite case the reasoning is essentially analogous. Then, our purpose is to reach at T the branch of the center manifold with movement to the initial fixed point (k_1^*, c_1^*). It is necessary to take into account that for the interval $(0, T)$ the dynamical system has a unique stationary solution (k_2^*, c_2^*). So, the problem is to find an initial condition $(k_T(0), c_T(0))$, depending on the extent of T, such that following a trajectory in the phase plane of the dynamical system with $a = a_2$, after the time elapses, the levels of consumption and capital at T are on the center manifold $W^c(k_1^*, c_1^*)$.

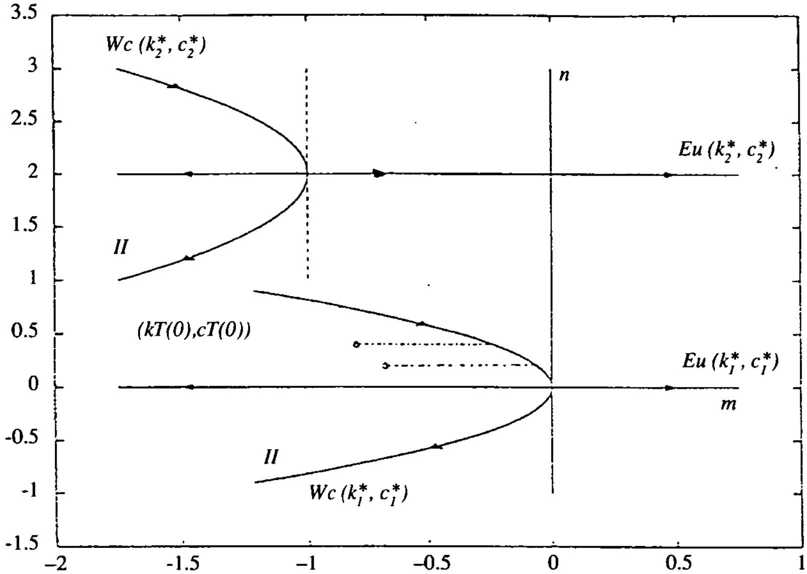

FIG. 4

Remembering a previons figure 2 we can assert that the several actions necessary in order to reach the objective are not possible if $i'''(k_1^*) = i'''(k_2^*) > 0$. But when $i'''(k_1^*)= i'''(k_2^*) < 0$ the purpose can be attained. The figure 4 shows initial conditions $(k_T(0), c_T(0))$ that allow to reach the optimal solution at T, following trajectories of the dynamical system when $a = a_2$. Let us remember that at T the unique fixed point is (k_1^*, c_1^*) again, because the parameter a takes value a_1 at this particular time.

The second-order approximation of the original dynamical system does not allow us to find the value of this point. But if we consider a first-order approximation and the condition on T, a straightforward calculation shows that consumption, $c_T(0)$, and capital, $k_T(0)$ must satisfy:

6 This assumption allows us to continue working with the S-shaped interest function choosed in the previous section.

$$c_T(0) = c(T),$$

$$k_T(0) = k_2^* + \frac{1}{re^{rT}} (c(T) - c_1^* \left[\frac{1}{r} \alpha_1^{(2)} (c(T) - c_1^* + 1 \right] - (c(T) - c_2^*) (1 - e^{rT}).$$

Thus, according with these relations in order that the consumer cane able to reach $W^c(k_1^*, c_1^*)$ in the branch converging to the fixed point (k_1^*, c_1^*), it will be necessary that the consumption satisfies $c(T) > c_1^*$ and also $c_T(0) > c_1^*$. That is, for the time that the fixed point (k_2^*, c_2^*) is active, the consumption stays constant and the capital grows. In long term, the consumption decreases to $c_1^* = c_2^* (a_2 - a_1)$ ant the capital must go back to the initial condition $k_1^* = k_2^*$.

5. CONCLUSIONS

In this paper we have analyzed the effects of a change in wages on capital and consumption, when these variables satisfy a dynamical system which is obtained from neccesary conditions of an intertemporal optimization model. Two kinds of disturbances has been considered: permanent and temporary change in wages. Under the first assumption, in the short-term, the Harberger-Laursen-Metzler effects are satisfied and the capital response to a decrease of wages is a falling-off. In long-term the capital increases due to the effect that the consumption has on it.

On the other hand, if we consider a temporary change of the parameter, following the phase plane when we assume that the wages grow, the consumption in the short-term is constant, whereas it always increases in the long-term. First capital increases and later, in a long-term analysis, decreases until it reaches the new stationary situation.

The papers of Pitchford and Sen and Turnovsky pointed out already that the results can be very different from the Harberger-Laursen-Metzler effect by means of a linear analysis. In this paper besides studying whether Harberger-Laursen-Metzler effect can be verified on the proposed model, we abbord a non-linear analysis.

BIBLIOGRAPHY

Guckenheimer, J. and Holmes, P. (1983): *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer-Verlag. New York.

Kamien, M. I. and Schwartz, N. L. (1981): *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*. North-Holland. New York.

Mizrach, B. (1992): «The state of economic dynamics: A review essay». *Journal of Economic Dynamics and Control*. Vol. 16, nº 1, pp. 175-190.

Pitchford, J. (1989): «Optimum Borrowing and the Current Account when there are fluctuations in Income». *Journal of International Economics*. Vol. 26, pp. 345-358.

Pitchford, J. (1991): «Optimum responses of the current account when income is uncertain». *Journal of Economic Dynamics and Control*, nº 15, pp. 285-296.

Sen, P. and Turnovsky, S. (1989): «Deterioration of the terms of Trade and Capital Accumulation: A re-examination of the Laursen-Metzler Effect». *Journal of International Economics*. Vol. 26, pp. 227-250.

Silberberg, E. (1990): *The Structure of Economics: A Mathematical Analysis*. Mac Graw-Hill. New York.