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EXISTENCE OF MATRICES WITH PRESCRIBED ENTRIES(*)

by

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1.- Introduction

Many papers have been written (see references) about the following

PROBLEM.- *Let* $\mathbb{F}$ *be an arbitrary field and* $a_1,\dots,a_n,c_1,\dots,c_n$ *be* n *elements of* $\mathbb{F}$. *Let* $(i_1,j_1),\dots,(i_n,j_n)$ *be* n *distinct positions of an* nxn *matrix. Construct an* n-*square matrix over* $\mathbb{F}$ *with characteristic polynomial* $f(x)=x^n-c_1x^{n-1}-\dots-c_n$ *and with prescribed entries* a_t *in the position* (i_t,j_t), $t=1,\dots,n$.

It is known ([1],[2],[6],[7-9]) that this problem in some exceptional cases does not have solution (i.e., there is no matrix satisfying the prescribed conditions). Namely, in [2] and [8] it is proved that if the prescribed positions are on the main diagonal, the problem has a solution if and only if the sum of the prescribed entries is equal to c_1, and if the prescribed positions are all in a row or column and the non-diagonal ones are required to be zero, the problem has a solution if and only if the entry prescribed for the diagonal position is a root of f(x).

2.- Main Result

In this paper we give a complete solution of the above problem. The main result is a theorem with a very long proof:

THEOREM.- *Apart from the exceptions listed below, there always exists an* nxn *matrix* A *over* $\mathbb{F}$ *with characteristic polynomial* f(x) *and with the entries* a_t *in the positions* (i_t,j_t), $t=1,\dots,n$.

The exceptions are:

(i) All entries on the main diagonal are prescribed and their sum is not c_1.

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(ii) There exists a row (or a column) all whose entries off the main diagonal are prescribed as zero, and the one on the main diagonal is prescribed and is not a root of $f(x)$.

(iii) There exists a row (or a column) all of whose entries off the main diagonal are prescribed as zero, and $f(x)$ *has no root in* $\mathbb{F}$.

(iv) $n=2$, *the prescribed positions are* $(1,2)$ *and* $(2,1)$, *and the equation* $x^2-c_1x-c_2+a_1a_2=0$ *has no solution in* $\mathbb{F}$.

(v) $n=3$, σ *is a circular permutation of order three, the prescribed positions have the form* $(i,\sigma(i)), i=1,2,3$, $a_1=a_2=a_3=0$, *and for every* $h\in\mathbb{F}$, $f(x)+h$ *does not have three roots in* $\mathbb{F}$.

(vi) $n=3$, $\mathbb{F}$ *has characteristic* 2, σ *is a permutation of order three which can be decomposed as a product of two disjoint cycles, the prescribed positions have the form* $(i,\sigma(i)), i=1,2,3$, *the entry* c_1 *is prescribed for a principal position, the other two are prescribed as zero, and* $f(x)$ *has no root in* $\mathbb{F}$.

(vii) $n=3$, *the prescribed positions are, unless we make a permutation,* $(1,2),(1,3),(2,3)$, *the prescribed entries are zero, and* $f(x)$ *has not all its roots in* $\mathbb{F}$.

(viii) $n=4$, *the prescribed positions are, unless we make a permutation,* $(1,3),(1,4),(2,3),(2,4)$, *the prescribed entries are zero, and* $f(x)$ *cannot be factorized in* $\mathbb{F}$ *as a product of two quadratic polynomials.*

Moreover, if the following condition is not satisfied:

$n=2$, $f(x)=(x-a)^2$, $a\in\mathbb{F}$, *the prescribed positions are* $(1,2)$ *and* $(2,1)$ *and* $a_1=a_2=0$,

and if none of the above exceptions occurs, then the matrix A *exists and can be chosen as nonderogatory.*

G. N. de Oliveira has proved in [7-9] that, for $n\geq 3$ and apart from the two exceptional cases exposed in the introduction, there always exists an nxn matrix with n prescribed entries and prescribed spectrum. This result has been improved by D. Hershkowitz who in [4] proves that the number of prescribed entries can be increased to 2n-3 without increasing the number of exceptions. In this paper we show that it is impossible to increase the number of prescribed positions to 2n-4 and obtain a matrix with an arbitrary prescribed characteristic polynomial. However, a question remains unanswered: If $n\geq 5$, then apart from the exceptional cases (i),

(ii) and (iii) of the Theorem, does there always exist an nxn matrix having 2n-5 entries and its characteristic polynomial prescribed ?.

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THIS PAPER IS TO APPEAR IN LINEAR ALGEBRA AND ITS APPLICATIONS

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