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TRANSITIVITY OF INTERSECTION PROPERTIES OF BALLS

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Subspaces of Banach spaces satisfying certain intersection properties of balls have been the subject of considerable attention. In [3] the reader will find a clarifying discussion of some of the main results.

Let $n \in \mathbb{N}$ be fixed and let X be a closed subspace of a Banach space Y. X is said to satisfy the *n-ball property* in Y if, given n closed balls $B_1, B_2, \ldots, B_n$ such that $X \cap B_k \neq \emptyset$ for $1 \leq k \leq n$, and $\bigcap_{k=1}^{n} B_k$ has nonempty interior, then $X \cap (\bigcap_{k=1}^{n} B_k) \neq \emptyset$. Alfsen and Effros [1] used n-ball properties to give a nice characterization of M-ideals. Let us recall [2] that X is called a *semi-L-summand* of Y if each element y in Y has a unique best approximation in X, say $\pi(y)$, and the metric projection π satisfies $\|y\| = \|\pi(y)\| + \|y - \pi(y)\|$ for all y in Y. When π is linear X is called an *L-summand* of Y. Also X is called a *semi-M-ideal* (resp. an *M-ideal*) of Y whenever X^{o} is a semi-L-summand (resp. an L-summand) of the dual space Y'. The result by Alfsen and Effros (see also [2; Theorem 2.9]) reads that X is an M-ideal of Y if and only if X has the 3-ball property in Y, if and only if X has the n-ball property in Y for all n in $\mathbb{N}$. Also Lima [2; Theorem 6.10] has shown that X is a semi-M-ideal of Y if and only if X has the 2-ball property in Y. Thus only the cases $n = 2,3$ are relevant in the definition of the n-ball property. It is well known that there are semi-M-ideals which are not M-ideals.

D. Yost has introduced a weaker intersection property. X is said to satisfy the $1\frac{1}{2}$-ball property in Y if, given a closed ball B_1 with centre at a point in X and another closed ball B_2 such that $X \cap B_2 \neq \emptyset$ and $B_1 \cap B_2$ has nonempty interior, then $X \cap B_1 \cap B_2 \neq \emptyset$. Semi-L-summands and semi-M-ideals satisfy the $1\frac{1}{2}$-ball property. In fact [3; Theorem 3] X has the $1\frac{1}{2}$-ball property in Y if and only if X^{o} has the $1\frac{1}{2}$-ball property in Y'.

Let us consider the following transitivity problem. If X has the m-ball property in Y and Y has the n-ball property in Z, where m,n are positive integers or $1\frac{1}{2}$, what can we say of X as a subspace of Z?
It is known and easy to show that the 3-ball property is transitive while

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transitivity of the 2-ball property was an open question posed to the author by D. Yost. We solve it as a consequence of the following result. From now on X,Y,Z will be Banach spaces with $X\subseteq Y\subseteq Z$.

1. THEOREM. *Assume that X is a semi-L-summand (resp. an L-summand) of Z and that the quotient space Y/X is a semi-L-summand (resp. an L-summand) of Z/X. Then Y is a semi-L-summand (resp. an L-summand) of Z.*

2. COROLLARY. *Let n be a positive integer. If X has the n-ball property in Y and Y has the n-ball property in Z, then X has the n-ball property in Z.*

D. Yost (private communication) has shown to us that our proof of Theorem 1 can be easily modified to obtain the following.

3. THEOREM. *Assume that X is a semi-L-summand of Z and that Y/X satisfies the $1\frac{1}{2}$-ball property in Z/X. Then Y has the $1\frac{1}{2}$-ball property in Z.*

By dualization of the above result we obtain

4. COROLLARY. *Assume that X has the $1\frac{1}{2}$-ball property in Y and that Y has the 2-ball property in Z. Then X has the $1\frac{1}{2}$-ball property in Z.*

The results on transitivity of intersection properties of balls can now be summarized as follows.

5. THEOREM. *Let $m,n \in \{1\frac{1}{2},2,3\}$ and $k = \mathrm{Min}\,\{m,n\}$. Assume that X has the m-ball property in Y and that Y has the n-ball property in Z. Then X has the k-ball property in Z unless $n = 1\frac{1}{2}$.*

The following elementary example by D. Yost shows that the above result is the best possible.

6. EXAMPLE. Let Z be $\mathbb{R}^3$ provided with the norm

$$\|(a,b,c)\| = |a| + |b| + |c|$$

and let us consider the subspaces Y,X of Z given by

$$Y = \{(a,b,c) : a = b\} \ , \quad X = \{(a,b,c) \ : c = 2a = 2b\}.$$

Then X has the 3-ball property in Y and Y has the $1\frac{1}{2}$-ball property in Z, but X fails the $1\frac{1}{2}$-ball property in Z. In particular the $1\frac{1}{2}$-ball property is not transitive.

REFERENCES

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2. A. LIMA, *Intersection porperties of balls and subspaces in Banach spaces*. Trans. Amer. Math. Soc. 227(1977), 1-62.

3. D. YOST, *The n-ball properties in real and complex Banach spaces*. Math. Scand. 50 (1982), 100-110.