EXTRACTA MATHEMATICAE 1, 9-11 (1986)

APERY BASIS AND POLAR INVARIANTS OF PLANE CURVE SINGULARITIES

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For an irreducible plane algebroid (or analytic) curve over C, M. Merle proves in [3] a relationship between the polar invariants of C (introduced by B. Teissier in [4]), i.e. invariants relative to a general polar curve of C. The situation considered by Merle is more general. Thus, in this paper, we show, for irreducible plane algebroid curves over arbitrary algebraically closed fields, the relationship between certain numerical invariants (which correspond to the polar invariants in the complex case) and a certain type of plane algebroid curves.

Let k be an algebraically closed field. For and algebroid curve C over k we mean a reduced complete noetherian local ring (A,m) of Krull dimension 1 which contains k as a coefficient field and with $\dim_k(m/m^2) \leqslant 2$. C is irreducible if A is a domain. If $\{x,y\}$ is a basis of m, one has a surjective k-algebra homomorphism $p : k[[X,Y]] \rightarrow A$ with $p(X) = x$ $p(Y) = y$, (X,Y indeterminantes over k), so one has $A = k[[X,Y]]/\mathrm{Ker}\,p$, Ker p being a radical principal ideal. If $f \in k[[X,Y]]$ is a generator of Ker p, we will say that $f = 0$ is an equation for C. The multiplicity of C will be denoted by e(C).

From now on C, and hence f, will be irreducible. The integral closure $\overline{A}$ of A in its quotient field is a discrete valuation ring, whose valuation will be denoted by v and one has $v(\overline{g}) = (C,D)$ = intersection multiplicity of C and D, for each curve D with equation $g = 0$, and $\overline{g} = p(g) \in A$.

The semigroup of values $S = S(C)$ is the additive semigroup of natural numbers given by $S = \{v(\overline{g}) \mid \overline{g} \in A - \{0\}\}$. The minimal set of generators of S is defined to be the set $\overline{\beta}_o,\dots,\overline{\beta}_g$ where $\overline{\beta}_o = \min(S - \{0\})$, $\overline{\beta}_i = \min\{a \in S \mid a \notin \langle\overline{\beta}_o,\dots,\overline{\beta}_{i-1}\rangle\}$, and for each i, $\langle\overline{\beta}_o,\dots,\overline{\beta}_{i-1}\rangle$ denotes the semigroup generated by $\overline{\beta}_o,\dots,\overline{\beta}_{i-1}$.

(*) A.M.S. Subject Classification (1980): 14-B-05, 14-H-20, 32-B-30

If we set $e_i = (\bar{\beta}_o,\dots,\bar{\beta}_i)$ and $n_i = e_{i-1}/e_i$ $(i = 1,2,\dots,g)$ then one also has $\bar{\beta}_i = \min\{a \in S \mid e_{i-1} \nmid a\}$ and $e_o > e_1 > \dots > e_g = 1$ (see [2], 4.3.6 and 4.3.10).

Now take $d \in S - \{0\}$ and A_d the set of integers of type $\min(S \cap (j+d\mathbb{N}))$ for some j, $0 \leqslant j \leqslant d-1$. Order the elements of A_d in such a way that $A_d = \{a_o < a_1 < \dots < a_{d-1}\}$. The ordered set A_d is Known as the Apery basis of S relative to d. En [1] is proved that if $d = \bar{\beta}_o$ then

$$a_{\sum_{p=1}^{g} s_p(\bar{\beta}_o/e_{p-1})} = \sum_{i=1}^{g} s_p\bar{\beta}_g , \quad 0 \leqslant s_p < n_p, \quad p = 1,2,\dots,g.$$

Note that if $0 \leqslant i \leqslant \bar{\beta}_o-1$ then i has a unique writting of type

$$i = \sum_{p=1}^{g} s_p\left(\frac{\bar{\beta}_o}{e_{p-1}}\right) \quad \text{with} \quad 0 \leqslant s_p < n_p, \quad p = 1,2,\dots,g.$$

Our principal results are stated below. Assume $d = \bar{\beta}_o$.

<u>Theorem 1</u>.- Let D be an irreducible plane algebroid curve and $\bar{\beta}'_o,\dots,\bar{\beta}'_g$, the minimal set of generators of S(D) and $e'_i = (\bar{\beta}'_o,\dots,\bar{\beta}'_i)$. Then one has

a) If $(C,D)/e(D) > \bar{\beta}_i/(\bar{\beta}_o/e_{i-1})$ then $(\bar{\beta}_j/e_i) = (\bar{\beta}'_j/e'_i)$

$0 \leqslant j \leqslant i$

b) If $(C,D)/e(D) < \bar{\beta}_i/(\bar{\beta}_o/e_{i-1})$ then $(C,D) \in \langle \bar{\beta}_o,\dots,\bar{\beta}_{i-1}\rangle$.

<u>Theorem 2</u>.- Let D be an irreducible plane algebroid curve over k with equation $h = 0$ such that $0 < e(D) = i < \bar{\beta}_o-1$ and $(C,D) = a_i$. If

$i = \sum_{i=1}^{g} s_p(\bar{\beta}_o/e_{p-1})$ then $h = h_1\dots h_g$ and $h_p = \prod_{j=1}^{r_p} h_{p,j}$, $p=1,2,\dots,g$, with $h_{p,j} \in k[[X,Y]]$ irreducible verifying

a) $\sum_{j=1}^{r_p} e(D_{p,j}) = s_p(\bar{\beta}_o/e_{p-1}) \quad p = 1,2,\dots,g.$

b) $(C,D_{p,j})/e(D_{p,j}) = \bar{\beta}_p/(\bar{\beta}_o/e_{p-1}) \quad p = 1,2,\dots,g$

$j = 1,2,\dots,r_p$

The proof of 3.2 follows from 3.1, by using induction on $\sum_{p=1}^{g} s_p$. The proof of 3.1 is esentially based in the computation of (C,D) by means of Hamburger-Noether expansions, (see [2]). On the other hand the existence of curves D with e(D) = i, and (C,D) = a_i is guarantized by the existence of curves D_j with $e(D_j) = \frac{\bar{\beta}_o}{e_j}$, $(e, D_j) = \bar{\beta}_{j+1}$, j = 0,1,..,g ([2], 4.2).

FInally, we remark that if k = $\mathbb{C}$ then the polar invariants are given by

$$I(C) = \{ \bar{\beta}_o-1, \bar{\beta}_1-1, \ldots, \bar{\beta}_g/(\bar{\beta}_o/e_{g-1}) -1 \}$$

(see [3] and [4]). Moreover, the theorem 3.1 of [3] is a particular case of 3.2 since a general polar P(C) of C verifies e(P(C)) = n−1 and (C,P(C)) = a_{n-1}.

References

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