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ON INCOMPARABILITY OF BANACH SPACES

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Let X and Y be (infinite-dimensional real or complex) Banach spaces. X and Y are said to be totally incomparable if X and Y have no isomorphic subspaces of infinite-dimension.

H.P.Rosenthal introduced this concept in [2] (see also [1]) and gave a characterization of totally incomparable Banach spaces. A dual concept of incomparability is the following:

DEFINITION. X and Y are said to be totally coincomparable if they have no isomorphic quotients of infinite-dimension.

If X',Y' are the dual spaces of X,Y respectively, it is clear that if X' and Y' are totally incomparable (coincomparable), But the converses of these properties are not true. In fact, the sequence spaces ℓ_1 and ℓ_2 are totally incomparable but ℓ_∞ has a quotient isomorphic to ℓ_2, hence ℓ_2 and ℓ_∞ are not totally coincomparable. On the other hand, ℓ_∞ and c_0 are totally coincomparable because each separable quotient of ℓ_∞ is reflexive while every infinite-dimensional quotient of c_0 is isomorphic to a subspace of c_0 and then no reflexive

[1]; nevertheless $\ell_1 \subset \ell_1'' = \ell_\infty'$. Of course, reciprocals are valid if X and Y are reflexive.

H.P. Rosenthal 2 proved:

(I) X and Y are totally incomparable if and only if for every Banach space Z containing respective isomorphic copies M and N of X and Y the algebraic sum M+N is closed in Z.

In this paper, the main result is stated as follows:

(II) X and Y are totally coincomparable if and only if for every Banach space Z containing subspaces M and N such that $Z/M \simeq X$ and $Z/N \simeq Y$, the algebraic sum M+N is closed in Z.

Further, we give a proof of (I) that is interesting because is simpler than that in [2] and can be "dualized" regarding to show (II).

REMARKS. (1) It is easy to prove (II) for a finite number of Banach space in the same way that (I) in [2]. That is, if $M_1,\dots,M_k$ are subspaces of a Banach space Z such that $Z/M_1,\dots,Z/M_k$ are pairwise totally coincomparable, then $M_1+\dots+M_k$ is closed in Z. Also if R, S are disjoint subsets of $\{1,\dots,k\}$ then $Z/(\sum_{i\in R} M_i)$ and $Z/(\sum_{i\in S} M_i)$ are totally coincomparable.

(2) The result (II) give us information about the properties of the quotients of a Banach space from the relative position of the subspaces which generate then. For example, if M and N are subspaces of a Banach space X such that M+N is not closed and $X/M \simeq \ell_2$, then X/N has a quotient isomorphic to ℓ_2.

REFERENCES

[1] Lindenstrauss, J., Tzafriri, L.: Classical Banach spaces I, Berlin Heildelberg, Springer-Verlag, 1977.

[2] Rosenthal, H.P.: On totally incomparable Banach spaces, J. Functional Anal. 4 (1969), 167-175.