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ON THE ESSENTIAL ORDER SPECTRUM

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Let E be a complex Banach lattice and denote by $\mathcal{L}^r(E)$ the Banach algebra of regular operators in E. In [7], Schaefer introduced the order spectrum $\sigma_o(T)$ as the spectrum of T in $\mathcal{L}^r(E)$, i.e. $\lambda \notin \sigma_o(T)$ iff $\lambda I - T$ has a regular inverse in E. More recently, Arendt and Sourour [1] introduced the essential order spectrum $\sigma_{oe}(T)$ as the spectrum of the equivalence class of T in $\mathcal{L}^r/\mathcal{K}^r$, where $\mathcal{K}^r$ is the closure of the finite dimensional operators in $\mathcal{L}^r(E)$. They show that $\lambda \notin \sigma_{oe}(T)$ iff there are finite dimensional operators F_1, F_2 in E and $S \in \mathcal{L}^r(E)$ such that

$$TS = I + F_1, \quad ST = I + F_2.$$

The essential order spectrum is of course analogous to the usual essential spectrum $\sigma_e(T)$ of T and we consider here two questions concerning this analogy:

1. THE DISTRIBUTION OF $\sigma_o(T)\setminus\sigma_{oe}(T)$

It is well known that $\sigma(T)\setminus\sigma_e(T)$ is the union of isolated points and some holes in $\sigma_e(T)$. The following theorem shows that the same is true for $\sigma_o(T)\setminus\sigma_{oe}(T)$. This answers a question of Arendt and Sourour in [1].

Theorem 1: Let T be a regular operator on a complex Banach lattice. For a component C of $\mathbb{C} - \sigma_{oe}(T)$ we have either

i) $C \cap (\mathbb{C} - \sigma(T)) = \emptyset$ and $C \subset \sigma_o$, or

ii) $C \cap (\mathbb{C} - \sigma(T)) \neq \emptyset$. Then C contains at most countable many isolated points of $\sigma_o(T)$.

Let E be a Banach function space on a measure space (Ω,μ) without atoms. We say that $T \in \mathcal{L}^r(E)$ has an atomic representation if T is of the form

$$Tf(y) = \Sigma\, a_n(y) f(\sigma_n(y))$$

with measurable maps $a_n : \Omega \to \mathbb{C}$, $\sigma_n : \Omega \to \Omega$. For such an operator one can show that $\sigma_o(T) = \sigma_{oe}(T)$ but we have examples with $\sigma_e(T) \neq \sigma_{eo}(T)$.

2. CONDITIONS FOR $r_e(T) = r_{oe}(T)$

Denote by r, r_o, r_e and r_{oe} the radius of the spectra $\sigma(T)$, $\sigma_o(T)$, $\sigma_e(T)$ and $\sigma_{oe}(T)$, respectively.

In [1] A2a, Arendt and Sourour give an example of a compact, positive convolution operator T with $r_{oe} > 0$. On the other hand there are important classes of positive operators for which $r_e(T) = r_{eo}(T)$ still holds.

Theorem 2: Let E be a Banach function space such that E, E' have order continuous norm, i.e. E is reflexive.
Then we have $r_e(T) = r_{eo}(T)$ for every positive integral operator $T: E \to E$.

Remark 1: Via representation theorems one can reformulate Theorem 1 for Banach lattices E and operators T in the band generated by $E' \otimes E$ in $\mathcal{L}^r(E)$.
If E is purely atomic (e.g. a sequence space with unconditional basis) then $(E' \otimes E)^{\perp\perp} = \mathcal{L}^r(E)$ and the result holds for all positive operators $T: E \to E$.

Remark 2: Under the assumptions of Theorem 2 we have in particular that $r_e(T) \in \sigma(T)$. For purely atomic Banach lattices this was first observed by Caselles [2]. de Pagter and Schep [4] showed recently that $r_e(T) \in \sigma(T)$ holds for positive operators which map order intervals into relatively compact sets. On the one hand this condition is more general than the assumption of Theorem 2, on the other hand the stronger conclusion $r_e(T) = r_{eo}(T)$ does not hold for this more general class (see the example quoted in the beginning of this section). But also $r_e(T) \in \sigma(T)$ is not true for an arbitrary positive operator as shown by the next example. A similar example was found independently by de Pagter and Schep [4].

Example: Consider the positive measure μ on $[0,2\pi]$ given by the Riesz-product

$$d\mu = \prod_{j=1}^{\infty} [1 - \sin(4^j x)]dx$$

By direct calculation or by comparison with [3], p. 197, one can show that its Fourier transform is given by

$$\hat{\mu}(m) = \begin{cases} 1 & \text{if } m = 0 \\ (\frac{i}{2})^k & \text{if } m = \pm 4^{j(1)} \pm \ldots \pm 4^{j(k)}, \quad 1 < j(1) < \ldots < j(k) \\ 0 & \text{otherwise .} \end{cases}$$

(In the notation of [3] we have $\Theta_j = 4^j$, $a(\Theta_j) = \frac{i}{2}$.) The convolution operator $T_\mu : L_2(T) \to L_2(T)$, $T_\mu = f * \mu$, has

$$\sigma(T_\mu) = \{\hat{\mu}(m) : m \in \mathbb{Z}\}.$$

Therefore $r_e(T_\mu) = \frac{1}{2} \notin \sigma(T_\mu)$ and by [5], Corollary 2.7, the same is true if we consider T_μ as an endomorphism of $L_p(T)$, $1 < p < \infty$.

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