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INTERLACING INEQUALITIES FOR GENERALIZED SINGULAR VALUES (*)

by

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1.-INTRODUCTION

Let A be an mxn complex matrix . The singular values

(1) $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_{\min(m,n)}(A)$

of A are the common eigenvalues of the positive semidefinite matrices $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$.

Since AA^* is m-square and A^*A is n-square, the eigenvalues of $(AA^*)^{1/2}$ and $(A^*A)^{1/2}$ do not coincide in full . However, it is well known that the nonzero eigenvalues (including multiplicities) of these two matrices allways coincide.

It is convenient to define $\sigma_t(C)$, $C \in \mathbb{C}^{mxn}$, to be zero for $t \geq \min(m,n)$. This frees us from constantly indicating the variation of the indices in the theorems.

We consider the following theorem (see [1]and [5]) : Let A be an mxn complex matrix with singular values $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$. Let B be a (m-s)x(n-r) submatrix of A with singular values $\sigma_1(B) \geq \sigma_2(B) \geq \ldots$. Then $\sigma_k(A) \geq \sigma_k(B) \geq \sigma_{k+s+r}(A)$, $k = 1,2,\ldots$.

This paper deals essentially with the question of finding conditions under which this interlacing theorem for singular values survives a generalization (cf. [3]), in their definition; namely, the introduction of two arbitrary norms in the Courant-Fischer minimax characterization of singular values (see [4] , pag. 321).

Let A be an mxn complex matrix and let $p: \mathbb{C}^m \longrightarrow \mathbb{R}$, $q: \mathbb{C}^n \longrightarrow R$ be vector norms . The generalized singular va-

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lues of A are the non-negative numbers defined as

$$\alpha_k(A) = \min_{G_{n-k+1} \in \mathbb{G}_{n-k+1}} \max_{\substack{x \in G_{n-k+1} \\ x \neq 0}} \frac{p(Ax)}{q(x)}$$

where $\mathbb{G}_{n-k+1}$ is the set of all subspaces of $\mathbb{C}^n$ of dimension $n-k+1$, $k = 1,\dots,n$.

Also, we agree that $\alpha_i(C)$, $C \in \mathbb{C}^{m \times n}$, is zero if $i \geq \min(m,n)$. If p , q are the Euclidean norms, then $\alpha_k(A) = \sigma_k(A)$ (see [4] , pag. 321).

2.-RESULTS

We show that the generalized singular values (briefly p,q-s.v.) verify the well-known inequalities of Ky Fan for the sum of matrices (cf. [2]): Let A, B be mxn complex matrices, then the p,q-s.v. of A, B and A+B verify

$$\alpha_{k+j-1}(A+B) \leq \alpha_k(A) + \alpha_j(B) \quad .$$

If $u = (u_1,\dots,u_{n_1})^T \in \mathbb{C}^{n_1}$ and $v = (v_1,\dots,v_{n_2})^T \in \mathbb{C}^{n_2}$ write (u,v) for the vector $(u_1,\dots,u_{n_1},v_1,\dots,v_{n_2})^T \in \mathbb{C}^{n_1+n_2}$. The main result is the following theorem:

Theorem.- Let A be an mxn complex matrix and let B be a $(m-s)x(n-r)$ submatrix of A obtained by eliminating the s last rows and r last columns of A . Let $\pi(x) := p(x,0)$ be the norm induced by p in $\mathbb{C}^{m-s}$, and let $\chi(z) := q(z,0)$ be the norm induced by q in $\mathbb{C}^{n-r}$. Then

$$\alpha_k(B) \geq \alpha_{k+s+r}(A) \quad ,$$

$\alpha_k(B)$ being the π,χ -s.v. of B .

Suppose that the conditions

$$\pi(x) \leq p(x,y) \quad \text{for all} \quad x \in \mathbb{C}^{m-s} \ , \ y \in \mathbb{C}^s$$
$$\chi(z) \leq q(z,w) \quad \text{for all} \quad z \in \mathbb{C}^{n-r} \ , \ w \in \mathbb{C}^r$$

are satisfied . Then

$$\alpha_k(A) \geq \alpha_k(B) \quad .$$

The Hölder norms verify the hypothesis of de above result . We say that a norm $\mu: \mathbb{C}^k \longrightarrow \mathbb{R}$ is symmetric when $\mu(x_1,\dots,x_k)$ is unchanged if $(x_1,\dots,x_k)$ is replaced by $(x_{\sigma(1)},\dots,x_{\sigma(k)})$, σ being any permutation of $\{1,2,\dots,k\}$. Note that if p and q are symmetric norms, this Theorem holds for any submatrix B of A, not necessarily that of the upper left corner.

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