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ON THE DEPENDENCE OF KOSZUL HOMOLOGY FROM THE GENERATORS OF AN IDEAL

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Let A be a commutative ring with 1 and let $x_1,\dots,x_n$ be a sequence of elements in A. We have the associated (homological) Koszul complex $K(x_1,\dots,x_n)$, whose homology groups we will denote $H_i(x_1,\dots,x_n)$ or $H_i(\underline{x})$.

The aim of this note is to state some results around the following question:

If the ideal I of A has generator sets $x_1,\dots,x_n$ and $y_1,\dots,y_m$, how are related $H.(\underline{x})$ and $H.(\underline{y})$?. In other words: can we obtain invariants of I through Koszul homology?. There is a known easy answer when A is a local ring, because two minimal generator sets of I have the same cardinality n=m and we pass from $\underline{x}$ to $\underline{y}$ by means of an invertible n x n matrix, obtaining by functoriality isomorphic Koszul complexes and homology.

For general A and $I=(x_1,\dots,x_n)$ the "extreme" cases i=0, n are clear, since $H_o(\underline{x})=A/I$ and $H_n(\underline{x})=\mathrm{Ann}_A(I)$. Then the first interesting case is $H_1(\underline{x})$ with $n\geq 2$. To our knowledge, the only existent result is due to Simis:

<u>Proposition 1 ([2])</u>. Let A be a commutative ring, $I=(x_1,\dots,x_n)=(y_1,\dots,y_m)$ an ideal of A. Then there is an isomorphism of A/I-modules

$$H_1(\underline{x}) \oplus (A/I)^m \approx H_1(\underline{y}) \oplus (A/I)^n$$

For the others $H_i(\underline{x})$ we have proved

<u>Proposition 2</u>. Let A be a commutative ring, $I=(x_1,\dots,x_n)=(y_1,\dots,y_m)$ an ideal of A. For each $i\geq 0$ there is an isomorphism of A/I-modules

$$\bigoplus_{j=o}^{m} H_{i-j}(\underline{x})^{\binom{m}{j}} \approx \bigoplus_{k=o}^{n} H_{i-k}(\underline{y})^{\binom{n}{k}}$$

The cancellation problem posed by the above result can be solved in some cases. The most general we have reached is

Proposition 3. Let A be a commutative ring, $I=(x_1,\dots,x_n)=(y_1,\dots,y_n)$ an ideal of A such that A/I is semilocal noetherian. Then for every i

$$H_i(\underline{x}) \approx H_i(\underline{y})$$

More particular results are in fact consequences of proposition 1.

Corollary 1 ([4]). Let A be commutative ring, $I=(x_1,\dots,x_n)=(y_1,\dots,y_n)$ an ideal of A such that $H_1(\underline{x})=0$. Then $H_1(\underline{y})=0$.

Corollary 2. Let A be a commutative ring, $I=(x_1,\dots,x_n)=(y_1,\dots,y_{n+1})$ an ideal of A such that $H_1(\underline{x})=0$. Then $H_1(\underline{y})=A/I$.

Let Max A be the maximal spectrum of A. The strong cancellation theorems from [1], [3] give us

Corollary 3. Let A be a commutative ring, $I=(x_1,\dots,x_n)=(y_1,\dots,y_n)$ an ideal of a. if $n>\dim \text{Max}\ (A/I)$, then

$$H_1(\underline{x}) \approx H_1(\underline{y})$$

and the same is true if $n \geq \dim(A/I)$ when A is a finitely generated k-algebra over an algebraically closed field k.

Simis ([2]) suggest the possibility that the freeness of the A/I module $H_1(\underline{x})$ would be independent of the choice of generators for I. Our following example shows that this is not so.

Example 1. Let $B=R[X,Y,Z]/(X^2+Y^2+Z^2-1)$ the coordinate ring of the real sphere, $A=B[T]$, I the ideal of A generated by T. Denote by x,y,z the images of X,Y,Z in B. We have $I=(T)=(xT,yT,zT)$, $H_1(T)=0$ and

$$H_1(xT,yT,zT) \oplus A/I \approx (A/I)^3$$

but $H_1(xT,yT,zT)$ is a rang 2 projective non free B-module.

This example is minimal in some sense. Corollary 3 excludes a similar example over an algebraically closed field and corollary 2 one with only two generators. On the other hand, Bass cancellation theorem [1] excludes the use of more than three generators.

Example 1 answers in the negative the question of cancellation in proposition 1 with n=m, simply repeating twice T as a superflous generator. The following one uses only minimal sets of generators for I.

Example 2. With the same notations as in example 1, we have $I=(xT,yT,zT)=(xT,yT,(y+x+1)T)$ and

$$H_1(xT,yT,zT) \not\cong H_1(xT,yT,(y+x+1)T)$$

We close this note characterizing ideals which can be generated by A-sequences through Koszul homology.

Proposition 4. Let A be a commutative noetherian ring, $I=(x_1,\ldots,x_n)$ an ideal of A. I can be generated by some A-sequence if and only if $H_1(x_1,\ldots,x_n)$ is a stably free A/I-module with rang $n-\mu(I)$ (with $\mu(I)$=least number of generators of I).

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