EXTRACTA MATHEMATICAE 1, n.3, 103-114 (1986)

OPERATORS ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

Fernando Bombal
Facultad de Matemáticas
Universidad Complutense
28040 Madrid. Spain

The modern theory of linear bounded operators on spaces of continuous functions is greatly indebted to the work of A. Grothendieck, who in his fundamental paper |18| characterized several classes of operators on C(K) and, with his peculiar homological point of view, axiomatized several important properties of a Banach space in terms of the behaviour of the operators defined on it.

The following is an exposition of several results concerning to the study of some classes of operators on spaces of Banach valued continuous functions on compact Hausdorff spaces, following Grothendieck's directions.

Our terminology will be standard. But in order to prevent any doubt, let us fix some notation. Throughout the paper, E and F will be Banach spaces and K a compact Hausdorff space. C(K,E) will denote the Banach space of all continuous E-valued functions on K, under the supremum norm. When E is the scalar field, we shall write simply C(K). $\mathcal{L}(E,F)$ will stand for the space of all the operators (=linear bounded operators) between E and F. Recall that $T \in \mathcal{L}(E,F)$ is said to be

- -weakly compact if T maps bounded sets into relatively weakly compact sets.
- -Dunford-Pettis if T sends weakly convergent sequences into weakly convergent ones.
- -a Dieudonné operator if T transforms weakly Cauchy sequences into weakly convergent ones.
- -unconditionally converging if T maps weakly unconditiona-

lly Cauchy (w.u.c.) series into norm unconditionally convergent series.

It is easily seen that every weakly compact or Dunford-Pettis operator is a Dieudonné operator, and that every Dieudonné operator is unconditionally converging. No other non-trivial relations are true in general. Also, all these classes of operators are operator ideals, that is, they are stable under composition with arbitrary operators and, for a given pair or Banach spaces E and F, they are subspaces of $\mathcal{L}(E,F)$.

For a finitely additive measure m on some σ-algebra Σ, with values in $\mathcal{L}(E,F)$, let us denote by $\tilde{m}$ its semi-variation, i.e., the set function on Σ defined by

$$\tilde{m}(A)=\mathrm{Sup}\{\|\sum_i m(A_i)x_i\|\} ,$$

where the sup is taken over all the finite partitions of A in Σ and $\|x_i\|\leqslant 1$ (see |15|, p. 51). Recall that $\tilde{m}$ is said to be continuous at $\emptyset$ if $\tilde{m}(A_n)\to 0$ for every decreasing sequence $(A_n)\downarrow\emptyset$ in Σ or, equivalently, if there exists a control measure for m, that is, a positive countably additive measure ν on Σ such that $\lim\limits_{\nu(A)\to 0}\tilde{m}(A)=0$

1. <u>Some classes of operators on C(K,E).</u>

Given an operator T from C(K,E) into F, as a consequence of the Riesz representation theorem, there exists a finitely additive representating measure m of bounded semi-variation, defined on the Borel σ-field Bo(K) of K and with values in $\mathcal{L}(E,F^{**})$, in such a way that

$$(*)\quad T(f)=\int f\,dm \quad \text{for each} \quad f\in C(K,E) .$$

(see, for instance, |14|, p. 182). In particular, the dual space C(K,E)* is isometric to the space rcabv(Bo(K),E*) of all regular, countably additive E*-valued measures on Bo(K) of bounded variation, endowed with the variation norm.

The study of the relationship between an operator and its representating measure is one of the central topics in the theory. Let us remark that the expresion (*) makes sense for f belonging to the space B(Bo(K),E) of totally measurable E-valued functions on Bo(K) (|15|, p. 83), i.e., the uniform limits of Bo(K)-simple E-valued functions, and so it defines an extension $\hat{T}$ of T, which is simply the restriction to $\mathbb{B}$(Bo(K,E) of the bitranspose T** of T. It is easily seen (see |1|) that T is weakly compact if and only if so is T. In general, if $\mathcal{P}$ denotes either the class of weakly compact, Dunford-Pettis, Dieudonné or unconditionally converging operators, we have:

Theorem 1.1. (|4|) An operator T from C(K,E) into F belongs to the class $\mathcal{P}$ if and only if so is $\hat{T}$.

Note that for a non void Borel set $A \subset K$, the mapping

$$E \ni x \longrightarrow \theta_A(x) = \chi_A x$$

is an isometric embedding of E into $\mathbb{B}$(Bo(K),E), and for every operator $T:C(k,E) \longrightarrow F$ with representing measure m, the map $\hat{T}\,\theta_A$ is precisely m(A). Thus, if T belongs to $\mathcal{P}$, the same happens with m(A). Moreover, we have the following result (see |1|, |8|, |16|, |25|):

Proposition 1.2. Let $T:C(K,E) \longrightarrow F$ belong to $\mathcal{P}$ and let m be its representing measure. then

a) m is continuous at $\emptyset$.

b) For each $A \in Bo(K)$, m(A) maps E into F and belongs to $\mathcal{P}$.

However, conditions (a) and (b) do no characterize the operators between C(K,E) and F which belong to $\mathcal{P}$, as the following example shows:

Example 1.3. There exists a non unconditionally converging operator T from $C([0,1],c_0)$ into c_0 whose representing measure m verifies that $\tilde{m}$ is continuous at $\emptyset$ and for every Borel set in [0,1], m(A) is a compact operator from c_0 into itself. In fact, let λ be the Lebesgue measure on [0,1], and (r_n) a bounded sequence in C([0,1]) which is also an orthonormal system in $L^2(\lambda)$. Then (r_n) converges weakly to 0 in $L^2(\lambda)$ and therefore, also in $L^1(\lambda)$. For $f \in C([0,1],c_0)$ let us define

$$T(f)=\left(\int \langle f(t),e_n^* \rangle\, r_n(t)\,d\lambda(t)\right)_{n=1}^{\infty} \quad ,$$

where (e_n^*) are the functionals associated to the canonical bases (e_n) of c_o. Since (r_n) converges weakly to 0 in $L^1(\lambda)$, $T(f)$ belongs to c_o and it is easily seen that T is linear and bounded. The representing measure of T is given by

$$m(A)(x)=(\langle x,e_n^*\rangle \int_A r_n(t)d\lambda(t))_{n=1}^{\infty} \quad ,$$

for $A\in Bo(A)$ and $x\in c_o$. In consequence, $\|m(A)\| \leqslant \lambda(A)$, which proves that $\tilde{m}$ is continuous at $\emptyset$. Futhermore, m(A) is the limit in the norm operator topology of the sequence of finite rank operators (m_k), where

$$m_k(A)(x)=(\langle x,e_1^*\rangle \int_A r_1 d \quad , \ldots , \langle x,e_k^*\rangle \int_A r_k d \quad ,0,0,\ldots) \quad ,$$

and so m(A) is a compact operator. However, T is not unconditionally converging, because $\Sigma r_n e_n$ is a w.u.c. series in $C([0,1], c_o)$ but for every $n\in \mathbb{N}$ $\|T(r_n e_n)\|=1$.

In view of the preceding example, it is natural to ask when the conditions (a) and (b) of proposition 1.2 characterize the operators between C(K,E) and F belonging to the class $\mathcal{C}$. The rest of this section is devoted to give some answers to this question.

For reasons of brevity, we shall introduce the following notation: given a compact Hausdorff space K, we shall say that a Banach space E verifies condition $(W)_K$ (resp., $(DP)_K$, $(V)_K$, $(D)_K$) if the following holds:

For any Banach space F, an operator T from C(K,E) into F is weakly compact (resp., Dunford-Pettis, uncoditionally converging, Dieudonné) if and only if its representing measure m satisfies:

a) m is continuous at $\emptyset$.

b) For each Borel set $A\subset K$, m(A) maps E into F and is weakly compact (resp., Dunford-Pettis, unconditionallly converging, Dieudonné).

We shall omit the subindex K if E verifies the corresponding condition for all compact Hausdorff spaces K.

Theorem 1.4. (|23|) Let E be a Banach space. Then

1) E verifies (W) if and only if E* and E** have the Ra

don-Nikodym property.

2) E verifies (DP) if and only if E is a Schur space.

3) E verifies (U) if and only if E contains no isomorphic copy of c_o.

We should mention that the sufficiency of the above conditions was well known (see, f.i., |8| for (1) and (3), |16| for (2), etc,). Also C. Fierro had proved (1) in her Thesis.

For the property (D) we only know partial results:

Theorem 1.5. (|5|) Let E be a Banach space.

1) If E* has the Radon-Nikodym property, then E verifies (D) if and only if E** has the Radon-Nikodym property.

2) If E is weakly sequentially complete, then it verifies (D).

3) If E verifies (D), then it contains no isomorphic copy of c_o.

4) If E verifies (D) and E* fails to have the Radon-Nikodym property, then E contains an isomorphic copy of l^1.

Neither condition (2) is necessary nor conditions (3) and (4) are sufficient in order that a Banach space verifies (D). In fact, the James space J (see f.i. |27| p. 80) provides an example of a non weakly sequentially complete Banach space that verifies (D) (J* and J** being separable, they have the Radon-Nikodym property, and (1) applies). The James tree space X (see f.i. |27| p. 119) contains neither c_o nor l^1 and, since X* does not have the Radon-Nikodym property, it follows from (4) above that X does not verify (D). Finally, $X \oplus l^1$ provides a counter example to the sufficiency of (4).

Theorem 1.5 relies heavily on some recent results about the structure of C(K,E) that we shall refer later.

Another way of getting some answers to our question is to focus the attention on the compact space K. In this direction we have the following results:

Theorem 1.6. (|4|) If K is a compact Hausdorff dispersed space (i.e., K does not contain any perfect set), every Banach space verifies $(W)_K$, $(DP)_K$, $(U)_K$ and $(D)_K$.

The interesting thing is that theorem 1.6 characterizes in fact the compact dispersed spaces, as the following result shows.

Theorem 1.7. (|4|) For a compact Hausdorff space K, the following assertions are equivalent:

1) K is dispersed.
2) Every Banach space verifies $(W)_K$, $(DP)_K$, $(U)_K$ and $(D)_K$.
3) There exists a Banach space which contains a copy of c_o and verifies either $(W)_K$ or $(DP)_K$ or $(U)_K$ or $(D)_K$.

The proof of the only non trivial implication (3) $\Longrightarrow$ (1) is simply a reproduction of example 1.3 in an abstract setting. In fact, if K is not dispersed, there is a purely non atomic Radon probability sets of (|28|), theorem 2.8.10). Now we can construct a system of Borel sets of K, $\{A_i^n : 1 \leq i \leq 2^n , n \geq 0\}$ such that $A_1^o = K$, $A_1^n = A_{2i-1}^{n+1} \cup A_{2i}^{n+1}$; $\{A_i^n : 1 \leq i \leq 2^n\}$ is a partition of K for each n, and $\lambda(A_i^n) = 2^{-n}$ for $1 \leq i \leq 2^n$ and $n \geq 0$. Let (x_n) be a sequence in E equivalent to the canonical basis in c_o and choose $x_n^* \in E^*$ so that $\langle x_n, x_n^* \rangle = 1 = |x_n^*|$ for every $n \geq 0$. Let us write $r_n = \sum_{i=1} (-1)^i \chi A_i^n$ and define

$$T(f) = \left(\int_K \langle f(t), x_n^* \rangle r_n(t) d\lambda(t) \right)_{n=0}^{\infty}, \text{ for } f \in C(K,E) .$$

arguing as in example 1.3, we can prove that T is an operator from C(K,E) into c_o, whose representing measure m has λ as a control measure and such that m(A) is a compact operator for every Borel set A. However, T is not even unconditionally converging. In fact, $\sum r_n x_n$ is easily seen to be a w.u.c. series in $\mathbb{B}(Bo(K),E)$ but if T is the extension of T to $\mathbb{B}(Bo(K),E)$, then $\|T(r_n x_n)\| = 1$ for every $n \geq 0$, which proves that T is no unconditionally converging. From theorem 1.1 we conclude that T itself is not unconditionally converging.

Theorem 1.8. (|4|) For a compact Hausdorff space K, the following assertions are equivalent:

1) K is dispersed.
2) There exists a Banach space verifying $(W)_K$, whose dual lacks the Radon-Nikodym property.
3) There exists a Banach space verifying $(W)_K$, whose bidual lacks the Radon-Nikodym property.

4) There exists a Banach space verifying $(DP)_K$, which fails to have the Schur property.

2. Some properties of the Banach space C(K,E).

As we said at the beginning of this paper, several important properties of a Banach space are defined in terms of the behaviour of the operators on it, following the directions of the pioneering work of Grothendieck. Let us recall some of them: A Banach space E is said to have

- the Dunford-Pettis property (DPP is short) if every weakly compact operator on e is Dunford-Pettis.
- the Dieudonné property (DP in short) if every Dieudonné operator on E is weakly compact.
- Pelczynski's property V (VP in short) if every unconditionally converging operator on E is weakly compact.
- Grothendieck's property (GP in short) if every operator from E into c_o (or any separable Banach space) is weakly compact.

The DPP, DP and GP were introduced by Grothendieck, who proved also that every C(K) space has the two first properties and, when K is extremally disconnected, the GP too. Pelczynski defined the VP in |22| and proved that C(K) spaces have it. The obvious question is to ask what happens with the space C(K,E), when E is a Banach space. We shall try to give some answers to this question in the rest of the section.

All the above defined properties are inherited by finite products and complemented subspace. Since E is a complemented subspace of C(K,E), a necessary condition in order that C(K,E) has any of the mentioned properties is that E verifies it. But is it a sufficient condition? For the GP property the answer is negative, except for trivial cases, and can be obtained as a consequence of the following surprising result:

Theorem 2.1. (|10|) If K is an infinite compact Hausdorff space and E is an infinite dimensional Banach space, then C(K,E) contains a complemented copy of c_o.

The idea of the proof is very simple. In fact, using the Josefson-Nissenzweign theorem (see f.i. |13|), we can produce a pair of sequences $(x_n) \subset E$ and $(x_n^*) \subset E^*$, such that (x_n^*) is weak* convergent to 0 and $\|x_n\|=1=\langle x_n, x_n^*\rangle$ for every $n \geq 0$. On the other hand, since K is infinite, there exists an infinite sequence of non void pairwise disjoint open sets (G_n). Choose $t_n \in G_n$ and $\phi_n \in C(K)$ so that $0 \leq \phi_n \leq 1$, $\phi_n(t_n)=1$ and ϕ_n vanishes outside G_n. It is then obvious that the subspace

$$Y=\{\sum_1^{\infty} a_n \phi_n x_n : a=(a_n) \in c_o\}$$

is isometric to c_o, and the mapping given by

$$P(f)=\sum_1^{\infty} \langle f(t_n), x_n^* \rangle \phi_n x_n \quad , \quad f \in C(K,E)$$

is a continuous projection from C(K,E) onto Y.

Since no Grothendieck space can contain a non reflexive se parable infinite dimensional complemented subspace, from theorem 2.1 we can conclude:

Corollary 2.2. (|21|) If C(K,E) is a Grothendieck space, then either K is finite or E is finite dimensional.

For the other properties, the situation is completely different. For example, an inmediate consequence of theorem 1.6 is the following

Theorem 2.3. (|9|) Let K be a dispersed compact Hausdorff space and E a Banach space. then C(K,E) has the DPP (resp., DP, VP) if and only if so does E.

However, for a general compact space, only partial answers are known:

Theorem 2.4. (|19|) If E does not contain a copy of l^1, then C(K,E) has the DP for every compact Hausdorff space K.

Theorem 2.5. ((11|) If E does not contain a copy of l^1 and has property (u) (i.e., for every weakly Cauchy sequence (x_n), there exists a w.u.c. series $\sum y_n$, such that $\{x_n - \sum_{i=1}^{n} y_i\}$ converges weakly to 0), then C(K,E) has the VP for every compact Hausdorff space K.

The proofs of both theorems 2.4 and 2.5 use a continuous selection theorem for a multivalued function with values in a convenient subset of $E^{\mathbb{N}}$. Theorem 2.4 had been conjectured in |3|, where was proved that C(K,E) has the DP under the stronger assuption of E* having the Radon-Nikodym property.

As for the DPP, the first positive result is due to I. Dobrakov, who proved in |16| that for a Schur space E and any compact Hausdorff space K, C(K,E) has de DPP. The same result holds when E is an (AM) Banach lattice (|24|). Much more difficult is the fact, proved by J. Bourgain, that $C(K,L^1(\mu))$ has the DPP (see, f.i., |7|). In spite of all this positive results, the general answer is negative, In fact, in |26| M. Talagrand built a Banach space H such that

i) H and H* have unconditional basis.

ii) H* is a Schur space. In particular, H and H* have the DPP.

iii) C([0,1],H) does not have the DPP.

Of course, theorem 2.3 tells us that C(K,H) has the DPP for every dispersed compact space K. If K is a non dispersed compact space, it is known (|28|), theorem 2.4.2) that there exists a continuous function θ from K onto [0,1]. In consequence, the map $f \longrightarrow f\theta$ embeds isometrically C([0,1],H) as a closed subspace of C(K,E). Since C([0,1],H) does not have the DPP, there exists a Banach space F and a wakly compact operator T from C([0,1],h) into F which is not Dunford Pettis. If we should be able to extend T to a weakly compact operator $\tilde{T}$ from C(k,E) to F, then, as T would not be Dunford-Pettis clearly, we could conclude that C(K,H) does not have the DPP. It turns out that this extension can in fact be made. More generally, we can consider the following situation: Let K and S be compact Hausdorff spaces and θ a continuous function from K onto S. Hence, for every Banach space E, the mapping

$$C(S,E) \ni f \longrightarrow J_{\theta}(f)=f\theta \in C(K,E)$$

is an isometric embedding. Then the following result holds.

<u>Theorem 2.6</u>. (|6|) Let E,F be Banach spaces and K,S and θ as before. If T is an operator from C(S,E) into F with a representing measure m such that its semi-variation has a control measu

re λ, then there exists an operator $\overline{T}$ from C(K,E) into F so that $\|\overline{T}\|=\|T\|$, $\overline{T}\cdot\dot{J}_0=T$ and the semi-variation of the representing measure of $\overline{T}$ has a control measure μ satisfying $\theta(\mu)=\lambda$. Moreover, if T is weakly compact, so is $\overline{T}$.

This last result was proved in |2| under the assumption of E* having the Radon-Nikodym property. Since Talagrand's space H verifies this assumption, the argument sketched before theorem 2.6 can be applied to obtain the following:

Theorem 2.7. (|2|) Let K be a compact Hausdorff space. The following properties are equivalent:

1) K is dispersed.
2) If E is a Banach space with the DPP, so is C(K,E).
3) C(K,H) has the DPP.

Theorem 2.6 can be used to extend several other calsses of operators from C(S,E) to the whole of C(K,E). For details, we refer to |6|.

In view of Talagrand's example, it seems difficult to find large classes of Banach spaces E for which C(K,E) has the DPP for an arbitrary compact Hausdorff space K. In this direction, C. Nuñez has proved in |20| that H** does not have the DPP, and he has conjecture that if the bidual E** of a Banach space E has the DPP, then C(K,E) also has the DPP for every compact Hausdorff space K.

REFERENCES

|1| J. Batt and J. Berg, Linear bounded transformations on the space of continuous functions, Journ. Funct. Analysis, 4 (1964), 215-239.

|2| F. Bombal, Weakly compact operators on spaces of continuous vector valued functions, Proc. of the A.M.S., 97 (1986), 93-96.

|3| F. Bombal and P. Cembranos, The Dieudonné property on C(K,E), Trans. of the A.M.S., 285 (1984), 649-656.

|4| F. Bombal and P. Cembranos, Characterization of some classes of operators on spaces of vector valued continuous functions, Math. Proc. Cambr. Phil. Soc., 97 (1985), 137-146.

|5| F. Bombal and P. Cembranos, Dieudonné operators on C(K,E), to appear in Bull. Acad. Pol. Sci.

|6| F. Bombal and B. Rodríguez-Salinas, Some classes of operators on C(K,E). Extension and applications, Archiv der Math. 47 (1986), 55-65.

|7| J. Bourgain, New classes of $\mathcal{L}^p$ spaces, Lect. Notes in Math. no. 889, Springer, Berlin, 1981.

|8| J. Brooks and P. Lewis, Linear operators and vector measures, Trans. of the A.M.S., 192 (1974), 139-162.

|9| P. Cembranos, On Banach spaces of vector valued continuous functions, Bull. Austr. Math. Soc., (1983), 175-186.

|10| P. Cembranos, C(K,E) contains a complemented copy of c_o, Proc. of the A.M.S., 91 (1984) 556-558.

|11| P. Cembranos, N. Kalton, E. Saab and P. Saab, Pelzynski's property V on C(Ω,E) spaces, Math. Ann., 271 (1985), 91-97.

|12| J. Diestel, A survey of results related to the Dunford-Pettis property, Proceedings of the Conference on Integration, Topology and Geometry in Linear Spaces, Contemporary Math. Volume 2, A.M.S., Providence, R.I., 1980.

|13| J. Diestel, Sequences and Series in Banach spaces, Springer, Berlin, 1984.

|14| J. Diestel and J.J. Uhl Jr., Vector measures, American Mathematical Society's Mathematical Surveys Volume 15, Providence, P.I. 1977.

|15| N. Dinculeanu, Vector measures, Pergamon Press, Oxford, 1967.

|16| I. Dobrakov, On representation of linear operators on C_o(T,X), Czechoslovak Math. Journ., 21 (1971), 13-30.

|17| C. Fierro, Compacidad débil en espacios de funciones y medidas vectoriales, Doctoral Thesis, Madrid, 1980.

|18| A. Grothendieck, Sur les applications linéaires faiblement compacts d'espaces du type C(K), Canad. J. Of Math., 5 (1953), 129-173.

|19| N. Kalton, e. Saab and P. Saab, On the Dieudonné property for C(Ω,E) preprint.

|20| C. Núñez, A conjecture about the Dunford-Pettis property, preprint.

|21| S.S. Khurana, Grothendieck spaces, Illinois J. of Math., 22 (1978), 79-80.

|22| A. Pelczynski, Banach spaces on which every unconditionally converging operator is weakly compact, Bull. Acad. Pol. Sci., 10 (1962), 641-648.

|23| P. Saab, Weakly compact, unconditionally converging and Dunford-Pettis operators on spaces of vector-valued continuous functions, Math. Proc. Cambr. Phil. Soc., 95 (1984), 101-108.

|24| J. Sánchez, Operadores en retículos de Banach, Thesis, Madrid 1985.

|25| C. Schwartz, Unconditionally converging and Dunford-Pettis operators on $C_X(S)$, Studia Math., 57 (1976), 85-90.

|26| M. Talagrand, La proprieté de Dunford-Pettis dans C(K,E) et $L^1(E)$, Israel Journ. of Math., 44 (1983), 317-321.

|27| D. Van Dulst, Reflexive and superreflexive Banach spces, Mathematical Center Tracts no. 102, Amsterdam, 1978.

|28| H.E. Lacey, The isometric theory of classical Banach spaces, Springer, Berlin, 1974.