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On Δ-stable Schwartz spaces

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Y este para Atala, que hace mucho que se lo merecía.

The diametral dimension of a locally convex space is defined to be the set (see |4|): $\Delta(E) = \{x \in R^N: \forall U \in U(E) \ \exists V \in U(E): x_n \delta_n(V,U) \longrightarrow 0\}$, where U(E) denotes a fundamental system of neighborhoods of 0 in E, and $\delta_n(V,U)$ stands for the n^{th} Kolmogorov diameter of V with respect to U: $\delta_n(V,U) = \inf \{k \geqslant 0 : V \subseteq L + kU$; L a subspace of E of dimension $\leqslant n\}$.

A lcs is said a Schwartz space iff $l_\infty \subsetneq \Delta(E)$, and nuclear iff the sequence $(n)_{n \in N} \in \Delta(E)$.

Let us consider the following theorem of |6|:

(I) Let E be a Schwartz space such that $\Delta(E) = \Delta(ExE)$. Then either $\Delta(E) = l_\infty$, or there is a unique G_∞-space $\Lambda(P)$ with $\Delta(E) = \Delta(\Lambda(P))$; if in addition E is nuclear, then $\Lambda(P)$ is stable.

Recall that a lcs is said stable when ExE is isomorphic to E; in accordance with this, we call a lcs Δ-stable when $\Delta(ExE) = \Delta(E)$, and Δ-representable by a lcs F when $\Delta(E) = \Delta(F)$.

Schwartz spaces of maximal (= R^N) or minimal (= l_∞) diametral dimension are Δ-stables, but no other classes seem to be known; by extending some results of |9| we characterize the Δ-stable G_∞-spaces:

1. A G_∞-space $\Lambda(P)$ is Δ-stable iff it is stable, and this happens iff $\forall a \in P \ \exists b \in P : a_{2n} \leqslant kb_n$, for some $k > 0$.

In particular, the G_∞-space of (I) is always stable.

Later on we will consider the case of spaces of maximal diametral dimension. Concerning the Schwartz spaces of minimal diametral dimension we have:

2. (a) There is no metrizable spaces of minimal diametral dimension
(b) There is no G_∞-spaces of minimal diametral dimension.

Point (b) explains why the case $\Delta(E) = l_\infty$ is necessarily excluded in (I) -however a Köthe space can be constructed with diametral dimension l_∞, showing that |8. (3)| cannot be improved- and point (a) that for metrizable spaces only the second possibility needs to be considered.

We can extend (I) to general spaces by considering only the stable ($\sup_n x_{2n}|x_n < +\infty$) sequences in $\Delta(E)$. Define $\Delta_s(E)$ -resp. $\Delta_{sm}(E)$- as the set of stable -resp. monotone increasing stable- sequences in $\Delta(E)$. Then:

3. Let E be a Schwartz space. Then E is Δ_s-representable by a G_∞-space iff $\Delta(E) \neq l_\infty$.

In fact we can prove that even $\Delta_{sm}(E\times E) = \Delta_{sm}(E)$ for all Schwartz spaces. An interesting question which can also be formulated in (I) is when the G_∞-space can be chosen metrizable when E is metrizable: this may not happen ($E = R^N$), but Terzioglu proves in |10| that it is so when E satisfies property DN of Vogt.

The use of stable sequences gives a chance to deal with some other questions in diametral dimension; in |7| appears a problem which, after some manipulations, is: characterize those lcs such that $a \in \Delta(E)$ implies $a^2 \in \Delta(E)$. We have:

4. All Δ-stable spaces (or all stable sequences in $\Delta(E)$) satisfy the above property.

in the literature there appear some different definitions of diametral dimension. Let us call: $\bar{\Delta}(E) = \{x \in R^N : \forall U \in U(E)\ \exists V \in U(E): x_n^{-1}\, \delta_n(V,U) \longrightarrow 0\}$ and $\Delta^*(E) = \{x \in R^N: \forall U \in U(E)\ \exists V \in U(E) : \delta_n(V,U) \leq |x_n|\quad n \in N\}$. The first definition is in |3|, and the second can be found in |5|. The equivalence between $\Delta(E)$ and $\bar{\Delta}(E)$ is more or less clear. To the equivalence of $\bar{\Delta}(E)$ with $\Delta^*(E)$ we have:

5. Let E be a Δ-stable space. Then $\bar{\Delta}(E) = \Delta^*(E)$ iff E is a Schwartz space.

Relative to spaces of maximal diametral dimension we have:

6. If E is a metrizable space of maximal diametral dimension then E is a subspace of R^N.

Using this we give in |1| an elementary proof that the class Ω of Schwartz spaces of maximal diametral dimension introduced in |3| cannot be generated by an ideal of operators.

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(This paper contains the main results of |1| and |2|.)