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AN INTERACTIVE SEQUENTIAL APPROACH TO MULTICRITERIA DECISION MAKING.

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The frame in which the multicriteria decision-making problems are stated as the search of a representation of the decision-maker (DM) preferences by means of a real-value function, has shown too much requirements in a great deal of real problems. A more slight idea, is to look for vector value functions, that is, to search for necessary and sufficient conditions such that for a given preference binary relation $(\prec)$ defined on a set X, there is a function $f=(f_1,\ldots,f_k)$ with $f:X\longrightarrow \mathbb{R}^k$ such that for all $x,x'\in X$,

$$x\prec x' \text{ iff } f(x)<f(x') \qquad (1)$$

with the usual definition of $(<)^{(*)}$ on $\mathbb{R}^k$.

A well-known result given by Milgram, Birkhoff,..., has been generalized (Roberts |1979|) as follows: If X is finite and $(\prec)$ a strict partial order of dimension at most $k, k>1$, there are k functions $f_1,\ldots,f_k$ satisfying

$$x\prec x' \text{ iff } f_i(x)<f_i(x')(\forall i) \qquad (2)$$

If it is overlooked that X be finite, a representation theorem given by Ríos-Insua |1980| shows the existence of a function verifying (1).

Theorem. Let be $x\in X$ where X is a rectangular subset of $\mathbb{R}^n$ and let $(\prec)$ be a preference binary relation on X. Assume

a) $(X,\prec)$ is a strict partial order intersection of k strict weak orders $(X,{}_j\prec)$.

b) $x<x'$ imply $x\prec x'$.

(*) $f(x)<f(x')\Longleftrightarrow f_i(x)<f_i(x')(\forall i)$

c) If $x \underset{j}{\prec} x'$ and $x' \underset{j}{\prec} x''$ there are $a_j \in [0,1]$ such that

$$a_j x+(1-a_j)x'' \underset{j}{\sim} x' \text{ for all } j=1,\dots,k.$$

Then there is a continuous vector value function $f:X \longrightarrow \mathbb{R}^k$ satisfying (1). This theorem has been afterwards considered in Skulimowski |1985| and Ríos and Ríos-Insua |1986|. The latter considers a "nest theorem" as follows:

Theorem. Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a strictly increasing function. If $x' \in X \subset \mathbb{R}^n$ is such that $f(x') \in M(Z)$, where $Z=f(X) \subset \mathbb{R}^k$, then $x' \in M(X)$ and $M(X) \supset \mathcal{E}(X,f)$.

The set $M(X)$ will be the maximal set of X for the corres pondig strict partial order ($<$) on $\mathbb{R}^n$ (analogously $M(Z)$), and $\mathcal{E}(X,f)$ the efficient set of X for f, which we call, set of va lue efficient decisions (of first order).

These two theorems lead us to consider, in the successive steps of a finite hierarchical configuration of criteria, a re duction of the efficient set of decisions x, which may be cho sen by the DM, having then a convergent process, that is, the value efficient decisions set of a given order shall contain the one of upper order and so on.

This idea can be translated to the case under uncertainty considering the concept of utility efficient set. We have

Definition. Let be $X \subset \mathbb{R}^n$, $\mathcal{P}_X$ the class of all simple probability distributions (*) on X and $u:X \longrightarrow \mathbb{R}^k$ a vector utility function. $P_x \in \mathcal{P}_X$ is said to be efficient for the mapping u, if there is no $P_{x'} \in \mathcal{P}_X$ such that

$$E\, u(P_x) \leqslant E\, u(P_{x'})^{(**)}$$

The efficient set of $\mathcal{P}_X$ for u, will be designate by $\mathcal{E}(\mathcal{P}_X, u)$ and called utility efficient set (of first order). Note that this definition is a generalization of the one under certainty, and so it has also sense to consider the efficient set $\mathcal{E}(X,u)$ subset of X.

On other hand, if we call

$$U=\left\{u \in \mathbb{R}^k : u=E\, u(P_x), P_x \in \mathcal{P}_X\right\}.$$

(*)We note that we could consider a more extensive class of dis tributions.

(**) $x \leqslant x' \Longleftrightarrow x_i \leqslant x_i' \ (\forall i)$ and $x_j < x_j'$ for at least one j.

the above definition is equivalent to write

$$E\ u(P_x) \in M(U)$$

Now, we can state that the efficient elements of $\mathcal{P}_X$, are distributions over points of $\mathcal{E}(X,u)$.

Theorem. Let be $X \subset \mathbb{R}^n$, $\mathcal{P}_X$ and let $u: X \longrightarrow \mathbb{R}^k$ be a strictly increasing vector utility function. If $P_x \in \mathcal{P}_X$ is efficient for u, then the rewards of P_x are points of $\mathcal{E}(X,u)$.

The converse of this theorem is not true, and we can write that

$$\mathcal{P}_{\mathcal{E}(X,u)} \supset \mathcal{E}(\mathcal{P}_X, u)$$

Hence, the set of decisions of interest for the DM, given u, shall be $\mathcal{E}(\mathcal{P}_X, u)$, that will be called utility efficient set (of first order).

The above theorems together with its extension to the case under uncertainty (Ríos and Ríos-Insua, |1986|) permit an interactive approach to the multicriteria decision-making problem.

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