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ON THE NUMBER OF CONJUGACY CLASSES IN A FINITE GROUP

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In this work, we get new results relative to the conjugacy classes of a finite group G.

Let N a normal subgroup of G, π a set of prime numbers, r(G) (resp. $r^{\pi}(G)$) the number of conjugacy classes of elements (resp. π-elements) of G, $\pi(G)$ the set of all different primes dividing $|G|$, G_{π} the set of all π-elements of G, and $\bar{G} = G/N$. In this paper, we analize the number $r^{\pi}(G)$ through the local analysis of the number $r^{\pi}_{G}(gN)$ of conjugacy classes of π-elements of G which intersect the coset gN . Our aims are threefold: to obtain upper and lower bounds of the number $r^{\pi}(G)$ in terms of the numbers $r^{\pi}(G/N)$, $r^{\pi}(N)$ and $|G'|$, where G' denotes the derived subgroup of G; to get the residue class of $r^{\pi}(G)$, modulo the "best" number, given in terms of the primes dividing $|G|$; and finally, to analyze the conjugacy-vector $\Delta_G = (|C_G(g_1)|,\dots,|C_G(g_r)|)$ of G, supposed that G is disjoint union of the conjugacy classes $Cl_G(g_i)$ $i=1,\dots,r=r(G)$ and being $|C_G(g_1)| \geq \dots \geq |C_G(g_r)|$. The results obtained are useful both for the calculation of the conjugacy-vector of a finite group and for the classification of finite groups according to the number of conjugacy classes (see examples 11-13, into paragraph 4).

We have collected here some of the main inequalities and congruences which are proved. They are, among other results, the following:

I) $r^{\pi}(G) \leq r^{\pi}(G/N).r^{\pi}(N)$, (1)

and equality holds if and only if $C_{G/N}(gN) = C_G(g)N/N$ for each π-element g of G.

II) If j is an integer number coprime to $o(\bar{g})$, then the following equalities hold: $r^{\pi}_{G}(gN) = r^{\pi}_{G}(g^{j}N)$ and ${}_{\pi}\Delta^{G}_{gN} = {}_{\pi}\Delta^{G}_{g^{j}N}$,

where ${}_{\pi}\Delta^{G}_{gN}$ is defined similarly to Δ_G, for , $Cl_G(g_i)$ $i=1,\dots,r=r^{\pi}_{G}(gN)$, the conjugacy classes of π-elements of G which intersect the coset gN .

III) $r^{\pi}_{G}(gN) \leq r^{\pi}_{N_G(gN)}(N)$, supposed that $C_{\bar{G}}(\bar{g})$ is a π-group. In addition, equality holds iff $\bar{g} \in \bigcap_{z \in N_G(gN)_{\pi}} \overline{C_G(z)}$.

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IV) Let G a L_π-group and H a Hall π-subgroup of G. Then, the following is true:

a) $r^\pi(G) \geq (r^\pi(G/N)-1).(|N \cap H|/|[H,G] \cap N|) + r^\pi_G(N)$. (2)

In particular, putting $\pi = \pi(G)$ into (2), we get

$$r(G) \geq (r(G/N)-1).|NG'/G'| + r_G(N),$$

inequality that includes as a special case the one given by E.A. Bertram (cf. [1] Th. 3; or [2] Lemma (2.1)).

b) If HN/N is abelian, then $r^\pi(G) \leq r^\pi(G/N).r_H(H \cap N)$, (3)

where $r_H(H \cap N)$ denotes the number of conjugacy H-classes that compose the normal subgroup $H \cap N$ of H .

c) If $N \cap [H,G] = 1$, being $[H,G] = \{ [h,g] \mid h \in H,\ g \in G \}$, then

$$r^\pi(G) = r^\pi(G/N).|H \cap N|, \qquad (4)$$

(putting $\pi = \pi(G)$, (4) yields Rusin's result given in [11] prop.3).

Inequalities (1) and (3) generalize Gallagher's inequality:

$$r(G) \leq r(G/N).r(N), \quad \text{(cf.[5])}$$

which follows from them as a special case for $\pi = \pi(G)$.

Moreover, if G/N is an abelian π-group, then III) yields:

$$r^\pi(G) \leq |G/N|.r^\pi_G(N).$$

In general, we obtain $r(G) \leq (r(\bar{G})-|Z(\bar{G})|).r(N) + |Z(\bar{G})|.r_G(N)$.

V) The following congruences are obtained:

a) If s^π_g denotes the number of conjugacy N-classes of π-elements of N fixed by the automorphism $f_g: N \longrightarrow N$ defined by $f_g(x) = x^g$ for each $x \in N$, then $s^\pi_g \equiv r^\pi_G(gN) \pmod{d_{|N_G(gN)/(N\langle g\rangle)|}}$.

(For each positive integer number t, we define d_t = g.c.d.(p-1| p is a prime dividing t)).

b) $r^\pi_G(N) \equiv r^\pi(N) \pmod{d_{|G|}.d_{|\bar{G}|}}$.

c) $r^\pi(G) \equiv |G_\pi| \pmod{(d_{|G|}.d_{|G|_\pi})/\text{g.c.d.}(|G|,d_{|G|_\pi})}$.

d) $r^\pi(G) \equiv |G| \pmod{\delta^\pi_{|G|}}$, where $\delta^\pi_{|G|}$ is defined by $\delta^\pi_{|G|}$ = g.c.d. $(q_1^2-1,\dots,q_e^2-1,q_{e+1}-1,\dots,q_s-1)$, supposed that $\pi(G) = \{q_1,\dots,q_s\}$ and $\pi(G) \cap \pi = \{q_1,\dots,q_e\}$.

e) $r^\pi(G) \equiv r^\pi(G/N).r^\pi(N) \pmod{\text{l.c.m.}(\delta^\pi_{|G|},d_{|G|}.d_{|\bar{G}|})}$.

Congruences of the type b) or c) for $\pi = \pi(G)$, were obtained in a different way by Poland (cf.[10] prop. (3.9)) and Mann (cf.[8] (16)). Congruence d) generalizes Hirsch's congruence given in [7] for $\pi = \pi(G)$. Finally, congruence a) relates the number $r^\pi_G(gN)$ with the number s^π_g; naturally, the number s^π_g is of easier calculation than $r^\pi_G(gN)$.

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