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AN EXTENSION ON THE A-POSTERIORI ERROR ANALYSIS FOR THE FINITE ELEMENT METHOD

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1. INTRODUCTION

In the finite element method there is much need for techniques to compute reliable a-posteriori error estimates at reasonable cost. Such error estimators are not only important for an assesment of the reliability of the results, but provide also a means for adaptive optimization of finite element meshes. In recent years, BABUSKA and RHEINBOLDT developed a general formulation using bilinear forms on pairs of suitable Hilbert spaces (see [4]), and later, they together with some co-workers, have used that theory in many papers related with the subject (see e.g. [2],[3],[5],[6],[7],[8], [11]).

Here, we give an alternative proof of the main theorem of [4] and suggest a-posteriori error estimators which are very close to those in [4], but which improve the lower bound of the error.

2. PRELIMINAIRES

Let H_1 , H_2 be two real Hilbert spaces with inner products $<\cdot,\cdot>_{H_i}$; $i=1,2$, and corresponding norms.

Let B be a proper bilinear form on $H_1 \times H_2$ (cf.[4] : (2.11)) and $f \in H_2'$ a given linear functional on H_2. We are interested in finding $u_0 \in H_1$ such that

$$B(u_0,v) = f(v) \text{ ; for all } v \in H_2 \tag{2.1}$$

With respect to this, we have the result:

Proposition 2.1. *Suppose that B and f are as stated above. Then there exists a unique solution $u_0 \in H_1$ of (2.1). Moreover,*

$$||u_0||_{H_1} \leq \frac{1}{C_2}||f||_{H_2'} \tag{2.2}$$

where C_2 is the constant of coerciveness of B (cf.[4] : (2.11)).

Proof : *See [1] : Theorem 5.2.1*

Now, let $\hat{P}$ be a family of pairs $(\hat{V},V)$ each of which consists of finite dimensional subspaces $\hat{V} \subset H_1, V \subset H_2$. Then, the following result holds.

Proposition 2.2. *Let B be a uniformly $\hat{P}$-proper form on $H_1 \times H_2$ (cf. [4] : (2.14)) and $f \in H_2'$ a given functional. Let $u_0 \in H_1$ be the unique element*

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satisfying (2.1)-(2.2). Then, for any $(\hat{V}, V) \in \hat{P}$ *there exists a unique* $\hat{u}_0 \in \hat{V}$ *such that*

$$B(\hat{u}_0, v) = f(v) \ ; \ \textit{for all } v \in V \tag{2.3}$$

and

$$||u_0 - \hat{u}_0||_{H_1} \leq (1 + \frac{C_1}{\hat{C}_2}) \inf_{w \in \hat{V}} ||u_0 - w||_{H_1} \tag{2.4}$$

where C_1 *is the constant of continuity of* B *on* $H_1 \times H_2$ *(cf. [4] : (2.11)) and* $\hat{C}_2$ *is the constant of coerciveness of* B *on* $\hat{V} \times V$ *(cf. [4] : (2.11)).*
Proof : *See [1] : Theorem 6.2.1*

In the following, we shall assume that

$$H_0^{k_i}(\Omega) \subseteq H_i \subseteq H^{k_i}(\Omega) \ \ , \textit{with } k_1, k_2 \in \mathbb{N} \cup \{0\} \tag{2.5}$$

and that

$$||\cdot||_{H_i} = ||\cdot||_{H^{k_i}(\Omega)} \tag{2.6}$$

We consider $\psi = \{\phi_1, \cdots, \phi_M\} \subseteq H^{k_2}(\Omega)$, a partition of unity of the domain Ω.

Let us also consider *the overlap index* $\rho(\psi)$ *of* ψ *(cf. [4]).*

In addition to this, we define set partitions T of $\overline{\Omega}$ consisting of Lipschitzian subdomains; that is

$$T = \{\Omega_1, \cdots, \Omega_m\} \ \ , \Omega_l \subset \Omega$$

$$\partial\Omega_l \ \textit{Lipschitzian} \ ; \ \overline{\Omega} = \bigcup_{l=1}^{m} \overline{\Omega}_l \ ; \ \Omega_l \cap \Omega_j = \emptyset, \ l \neq j.$$

With each Ω_l we associate a positive number h_l representing some measure of the size of Ω_l .

3. THE ERROR ESTIMATORS

Our main result may be stated as follows

Proposition 3.1. *Assume that* $\hat{P}$ *is as stated above, that* B *is a uniformly* $\hat{P}$*-proper bilinear form on* $H_1 \times H_2$*, and that* $f \in H_2'$ *is a given functional. Let* $\hat{T}$ *be an admissible family of triples* (ψ, T, V) *(cf.[4]). Let* $u_0 \in H_1$ *be the unique solution of (2.1) and, for any* $(\psi, T, V) \in \hat{T}$ *and corresponding* $(\hat{V}, V) \in \hat{P}$ *consider the error* $e = u_0 - \hat{u}_0$*, where* $\hat{u}_0$ *is the unique solution of (2.3). Then, there exist positive constants*

$$D_1 \geq \frac{1}{C_1 \rho^{1/2}} \ \ , \ D_2 \leq \frac{K^{1/2}}{C_2}$$

such that

$$D_1 \eta \leq D_1 \hat{\eta} \leq ||e||_{H_1} \leq D_2 \eta \leq D_2 \hat{\eta} \tag{3.1}$$

with

$$\eta^2 = \sum_{j=1}^{M} \eta_j^2 \ ; \ \eta_j = \sup_{v \in H_2, v \neq 0} \frac{|B(e,\phi_j v)|}{||\phi_j v||_{H_2}} \tag{3.2}$$

$$\hat{\eta}^2 = \sum_{j=1}^{M} \hat{\eta}_j^2 \ ; \ \hat{\eta}_j = \sup_{v \in H_2^{j,0}} \frac{|B(e,v)|}{||v||_{H_2}} \tag{3.3}$$

where

$$H_2^{j,0} = \{v \in H_2 \ , \ v \neq 0 \ / \ supp \ v \ \subset \ supp^o \phi_j\}$$

and ρ is a constant such that $\rho(\psi) \leq \rho$, for all $(\psi, T, V) \in \hat{T}$.
Proof . *See [9], [10] : Proposition 3.3.*
Remark 3.1. The main theorem of [4] suggests $D_1\eta$ and $D_2\eta$ as the lower and upper bounds, respectively, of $||e||_{H_1}$. In this sense, our result given by proposition 3.1 improves at least, the lower error bound. It is important to note that $\hat{\eta}_j$ may also be defined by

$$\hat{\eta}_j = \sup_{v \in H_o^{k^*}(supp^o \phi_j)} \frac{|B(e,v)|}{||v||_{H_2}} \tag{3.4}$$

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