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ORDER RELATION IN QUADRATIC JORDAN RINGS AND A STRUCTURE THEOREM

by

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The aim of a previous paper (see [5]) was to define a binary relation in linear Jordan algebras without nonzero nilpotent elements which is a partial order. So we defined there $x \leq y$ if and only if $xy = x^2$, $x^2y = x^3 = xy^2$ and we proved that this relation was a partial order when the linear Jordan algebra satisfied an additional condition (that was satisfied by every JB-algebra as we saw in [6]). Also we noted that for a Jordan ring A^+ , where A is an associative ring (with $x.y = \frac{1}{2}(xy + yx)$ this relation coincides with the usual associative order (see [1] , [2] and [4]).

With this order we could characterize the direct products of (linear) Jordan division rings.

In the present paper we consider quadratic Jordan algebras over a commutative associative ring Φ , with identity element. We define a relation which is always a partial order. Moreover, when the quadratic algebra is A^+ , obtained from the associative algebra using $U_x y = xyx$, the order relation defined here coincides with the known.

If $\frac{1}{2} \in \Phi$ and $(J,.)$ is the linear Jordan algebra associated with the quadratic Jordan algebra $(J,U,1)$ (that is the linear Jordan algebra with the same Φ-module and the linear product . given by: $x.y = \frac{1}{2} V_x y$), the order relation over $(J,.)$ defined in [5] induces and is induced by the order relation defined over the quadratic algebra. In particular the relation defined in [5] is always a partial order and the condition there imposed is not necessary. This answers a question posed in [5] .

Finally, the definitions of hyperatomic and orthogonally complete are extended and the known structure result can be obtained again.

Throughout the paper Φ will denote a commutative associative ring with identity element and $(J,U,1)$ a quadratic Jordan unital algebra over Φ, that is, J is a Φ-module, 1 is an element of J and $U: J \longrightarrow End_{\Phi}(J,J)$ is a quadratic mapping $U: x \longmapsto U_x$ which satisfies:

(1) $U_1 = I$

(2) $U_{U_x y} = U_x U_y U_x$ ("fundamental formula")

(3) $U_x \{y x z\} = \{U_x y\ z\ x\}$ ("isotope formula")

(4) $\{x\ y\ 1\} = \{x\ 1\ y\} = \{1\ x\ y\}$ (where $\{x\ y\ z\} = U_{x,z}\, y = (U_{x+z} - U_x - U_z)(y)$)

and such that these remain valid under extensions of the ring of scalars.

ORDER RELATION

Definition. Let J a unital quadratic Jordan algebra. We say $x \leq y$, $x, y \in J$, if $V_x x = V_x y$ (with $V_x y = U_{x,1}\, y$) and $U_x x = U_x y = U_y x$.

An equivalent version of the above definition is given by the following:

Lemma. If $x, y \in J$, then $x \leq y$ if and only if $V_x (x-y) = U_x(x-y) = U_{x-y}\, x = 0$.

Theorem 1. The relation $\leq$ is a partial order over J, J without nilpotent els.

Theorem 2. Let $\frac{1}{2} \in \Phi$ and $(J,.)$ the linear algebra associated with J. Then $x \leq y$ in J if and only if $x \leq y$ in $(J,.)$.

Corollary. The relation $\leq$ defined in [5] is always a partial order in linear Jordan algebras.

A STRUCTURE THEOREM

Definitions: a) J is a quadratic Jordan division algebra if U_x is invertible for every x in J.

b) Two elements a,b in J are said orthogonal if $0 = U_a b = U_b a = V_a b$.

c) A subset S of J is said orthogonal if every two differents elements of S are orthogonal. J is said to be orthogonally complete if every orthogonal subset of J has a supremum for the above relation $\leq$.

d) An idempotent element e of a quadratic Jordan algebra J is said to be a hyperatom if satisfies: i) e is a division idempotent, that is, $U_e J$ is a division algebra, ii) $U_a U_e = U_e U_a$ for every a in J, and iii) if $a \leq e$, then $a = 0$ or $a = e$.

e) J is said to be hyperatomic if for every a in J, $a \neq 0$, there is a hyperatom e such that $U_e a \neq 0$.

Now we can characterize the direct product of quadratic Jordan division algebras.

THEOREM. Let $J = (J,U,1)$ a quadratic Jordan algebra. Then J is a direct product of quadratic Jordan division algebras if and only if J is hyperatomic and orthogonally complete with respect to the order relation $\leq$ above defined.

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