EXTRACTA MATHEMATICAE 2, n.2, 81-83 (1987)

"ON THE NUMBER OF CONJUGACY CLASSES OF π-ELEMENTS IN A FINITE GROUP"(*)

Antonio Vera López and Mª Concepción Larrea

Departamento de Matemáticas. Facultad de Ciencias. Universidad del País Vasco. Apartado 644. Bilbao. SPAIN.

In this work, the number of conjugacy classes of π-elements (resp. non π-elements) of G is analyzed in terms of the corresponding numbers of G/N and for each N normal subgroup of G.

In the following, G will denote a finite group, N a normal subgroup of G, $\bar{G}$ =G/N, π a set of prime numbers and g_π the π-part of the element $g\epsilon G$. For each non-empty subset S of G, we define $[S]_\pi = \{s\epsilon S \mid s \text{ is a } \pi\text{-element}\}$ and $[S]'_\pi =$ $= S - [S]_\pi$.Moreover, $r^\pi_G(S)$ (resp. $r'^\pi_G(S)$) denotes the number of conjugacy classes of π-elements (resp. non π-elements) of G that intersect S. Further, we write $r^\pi(G) = r^\pi_G(G)$, $r'^\pi(G) = r'^\pi_G(G)$, $r(G) = r^{\pi(G)}(G)$ and $r_G(S) = r^{\pi(G)}_G(S)$, where π denotes the set of all prime numbers dividing $|G|$.

In this work, for each normal subgroup N of G and each π-element $\bar{g}$ of G/N, the following inequalities are proved:

$$\text{i) } r^\pi_G(gN) \leq r^\pi_{N_G(gN)}(N). \tag{1}$$

$$\text{ii) } r'^\pi_G(gN) \leq r'^\pi_{N_G(gN)}(N). \tag{2}$$

Furthermore, the equality holds in (1), (resp. in (2)) if and only if $\bar{g}$ is an element of $A^\pi_{g,N}$ (resp. $B^\pi_{g,N}$), where $A^\pi_{g,N}$ and $B^\pi_{g,N}$ are defined by

$$A^\pi_{g,N} = \bigcap\{\overline{C_G(x_\pi n)} \mid x\epsilon N_G(gN),\ n\epsilon N,\ x_\pi n \text{ is a } \pi\text{-element and } \bar{x}_{\pi'}\epsilon\overline{C_G(x_\pi n)}\}$$

$$B^\pi_{g,N} = \bigcap\{\overline{C_G(x_\pi n)} \mid x\epsilon N_G(gN),\ n\epsilon N,\ x_\pi n \text{ is a non } \pi\text{-element and } x_{\pi'}\epsilon\overline{C_G(x_\pi n)}\}$$

(here, π' denotes the complementary of π).

iii) For each normal subgroup N of G we have:

$$r^\pi(G) \leq (r^\pi(G/N)-|[Z(G/N)]_\pi|).r^\pi(N) + |[Z(G/N)]_\pi|.r^\pi_G(N), \tag{3}$$

the inequality (3) is proved by using the local inequality (1) for each element g of G. Furthermore, the equality holds in (3), if and only if the following conditions are satisfied:

(*) This work has been supported by the University of the Basque Country

a) $\bar{g}\epsilon A^{\pi}_{g,N}$ for each $\bar{g}\epsilon[\bar{G}]_{\pi}$.

b) $C_{\bar{G}}(\bar{g}) \leq \bigcap_{n\epsilon[N]_{\pi}} \overline{C_G(n)}$ for each $\bar{g}\epsilon[\bar{G}]_{\pi}-Z(\bar{G})$.

In particular, if G/N has a central Hall π-subgroup HN/N, then (3) yields

$$r^{\pi}(G) \leq |[\bar{G}]_{\pi}|.r^{\pi}_{G}(N) = |\bar{H}|.r^{\pi}_{G}(N). \tag{4}$$

Moreover, the equality holds in (4) if and only if $\bar{H}$ is contained in

$$\bigcap \{ \overline{C_G(x_{\pi}n)} \mid x\epsilon G,\ n\epsilon N,\ x_{\pi}n\epsilon[G]_{\pi} \text{ and } \bar{x}_{\pi'}\epsilon\overline{C_G(x_{\pi}n)}\}.$$

On the other hand, by using the local inequalities (1) and (2), the following inequality is proved.

iv) For each normal subgroup N of G, we have:

$$r'^{\pi}(G) \leq (r^{\pi}(G/N)-|[Z(\bar{G})]_{\pi}|).r'^{\pi}(N) + |[Z(\bar{G})]_{\pi}|.r'^{\pi}_{G}(N) + r'^{\pi}(G/N).r(N), \tag{5}$$

having equality if and only if the following assertions are true:

c) $C_{\bar{G}}(\bar{g}) = \overline{C_G(gm)}$ and $Cl_N(m)^g = Cl_N(m)$ for any $\bar{g}\epsilon[\bar{G}]'_{\pi}$ and $m\epsilon N$.

d) $\bar{g}\epsilon B^{\pi}_{g,N}$ for each $\bar{g}\epsilon[\bar{G}]_{\pi}$

e) $C_{\bar{G}}(\bar{g}) \leq \bigcap_{n\epsilon[N]'_{\pi}} \overline{C_G(n)}$ for each $\bar{g}\epsilon[\bar{G}]_{\pi}-Z(\bar{G})$.

In particular, from (3) and (5) we get the following inequalities:

$$r^{\pi}(G) \leq r^{\pi}(G/N).r^{\pi}(N)-(r^{\pi}(N)-r^{\pi}_{G}(N)) \tag{6}$$

and

$$r(G)-r^{\pi}(G) \leq r(G/N)r(N) - r^{\pi}(G/N)r^{\pi}(N) - (r'^{\pi}(N)-r'^{\pi}_{G}(N)) \tag{7}$$

v) For each normal subgroup N of G, the following inequalities are true:

$$r^{\pi}(G) \leq |[\bar{G}]_{\pi}|.r^{\pi}_{G}(N) \tag{8}$$

$$r'^{\pi}(G) \leq |[\bar{G}]_{\pi}|r'^{\pi}_{G}(N) + (|\bar{G}|-|[\bar{G}]_{\pi}|)r_G(N) \tag{9}$$

having equality in (8) (resp. in (9)) if and only if $\bar{g}\epsilon A^{\pi}_{g,N}$ and $C_{N_G(gN)}(n) = C_G(n)$ for any $\bar{g}\epsilon[\bar{G}]_{\pi}$ and $n\epsilon[N]_{\pi}$ (resp. $\bar{g}\epsilon B^{\pi}_{g,N}$ and $C_{N_G(gN)}(n) = C_G(n)$ for any $\bar{g}\epsilon[\bar{G}]_{\pi}$ and $n\epsilon[N]'_{\pi}$, and $\bar{g}\epsilon B^{\pi(G)}_{g,N}$ and $C_{N_G(gN)}(n) = C_G(n)$ for any $\bar{g}\epsilon\bar{G}-[\bar{G}]_{\pi}$ and $n\epsilon N$).

Finally we obtain the following inequality

$$\text{vi)}\quad r^{\pi}(G) \leq |[\bar{G}]_{\pi}|r^{\pi}_{G}(N) + (r^{\pi}_{G}(N)-(|[\bar{G}]_{\pi}|r^{\pi}(N))/|\bar{G}|) + (r^{\pi}(\bar{G})-1-(|[\bar{G}]'_{\pi(G)-\pi}|(|[\bar{G}]_{\pi}|-1))/|\bar{G}|). \tag{10}$$

In addition, the equality holds if and only if the following conditions are satisfied 1) G/N is π-group 2) For each $\bar{y}\in\bar{G}$ and each $x\in G-N$ such that $[\bar{x},\bar{y}]=\bar{1}$, $\bar{y}$ is an element of $\bigcap_{n\in N}\overline{C_G(xn)}$ and 3) $C_G(xn)\cap[N]_\pi=1$ for any $n\in N$ and each element $\bar{x}\in\bigcup_{\bar{g}\in[\bar{G}]}(\bar{G}-C_{\bar{G}}(\bar{g}))$.

In particular, putting $\pi=\pi(G)$, (10) yields

$$r(G)\leq|G/N|r_G(N)-(r(N)-r_G(N))-(|G/N|-r(G/N)),$$

and when G/N is an abelian group, we have

$$r(G)\leq|G/N|r_G(N)-(r(N)-r_G(N)),$$

and the equality holds if and only if $G=C_G(g)N$ for each $g\in G-N$.

REFERENCES

[1] P.X. Gallagher "The number of conjugacy classes in a finite group" Math. Z. 118 (1970), 175–179.

[2] C.H. Sah "Automorphisms of finite groups" J. Algebra 10 (1968), 47–68.

[3] A. Vera-López and L. Ortíz de Elguea "On the number of conjugacy classes in a finite group" To appear in Journal of Algebra.

(AMS Class : 20D)

(To appear in Proceedings of the Royal Society of Edinburgh)