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SEMIPRIME TERNARY ALGEBRAS CONTAINING MINIMAL RIGHT IDEALS

Antonio Fernández López and Eulalia García Rus

Departamento de Algebra, Geometría y Topología. Facultad de Ciencias. Universidad de Málaga. 29080 MALAGA.

Let Φ be a commutative associative ring with 1. A unitary Φ-module with a trilinear map $(a,b,c) \to \langle a\, b\, c \rangle$ of $A \times A \times A$ into A is called a *ternary algebra* (of Hestenes [2]) if the identities

$$\langle\langle a\, b\, c \rangle d\, e \rangle = \langle a \langle d\, c\, b \rangle e \rangle = \langle a\, b \langle c\, d\, e \rangle\rangle$$

are satisfied for all $a,b,c,d,e \in A$. It is easy to verify that the complex vector space $M_{n\times m}(\mathbb{C})$ of all rectangular matrices is a complex ternary algebra for the triple product defined by $\langle a\, b\, c \rangle = a\,(b^*)^t\, c$

where $(b^*)^t$ is the conjugate transpose of the matrix b. A submodule I of A is called an *inner ideal* (*right ideal* respectively) if $\langle x\, A\, x \rangle \subset I$, $x \in I$ ($\langle I\, A\, A \rangle \subset I$ resp.). An *ideal* is a submodule B of A such that

$\langle B\, A\, A \rangle + \langle A\, A\, B \rangle + \langle A\, B\, A \rangle \subset I$. A ternary algebra A is said to be *simple* if $\langle A\, A\, A \rangle \neq 0$ and A contains no ideals $B \neq 0, A$. Simple finite dimensional ternary algebras over a field were classified by Loos in [3], who also determined [4] the structure of simple ternary algebras satisfying dcc on inner ideals. A ternary algebra A is called ***semiprime*** (*prime* resp.) if $\langle B\, A\, B \rangle = 0$ implies $B = 0$, B ideal of A ($\langle B\, A\, C \rangle = 0$ implies $B = 0$ or $C = 0$, B and C ideals of A respectively). The *socle* Soc(A) of a semiprime ternary algebra A is defined to be the sum of all minimal right ideals of A. The socle is an ideal of A that is a direct sum of simple ideals. Moreover, Soc(A) coincides with the sum of all minimal inner ideals (this last characterization is the notion of socle given by McCrimmon [5] for a non-degenerate Jordan system). For an ideal B of a semiprime ternary algebra A the *annihilator* of B is the largest ideal Ann(B) satisfying $\langle \mathrm{Ann}(B)\, A\, B \rangle = 0$. An ideal B is called *essential* if $\mathrm{Ann}(B) = 0$. We have proved ([1]) the following:

THEOREM 1. *A semiprime ternary algebra* A *has socle essential if and only if* A *is a subdirect sum of prime ternary algebras* A_i *with nonzero socle such that* A *contains* $\oplus\, \mathrm{Soc}(A_i)$.

THEOREM 2. *The prime ternary algebras* A *with nonzero socle are these described below* :

1) Let D be a division associative algebra and suppose that $\mathcal{P} = (X_1,X_2,g)$ and $Q = (Y_1,Y_2,h)$ are two pairs of dual vector spaces over D. A pair $T = (T_1,T_2)$ where $T_1 \in \hom_D(X_1,Y_1)$ and $T_2 \in \hom_D(X_2,Y_2)$ is said to be *continuous* if there exist $T_1^{\#} \in \hom_D(Y_2,X_2)$ and $T_2^{\#} \in \hom_D(Y_1,X_1)$ (necessarily unique) such that

$h(x_1T_1,y_2) = g(x_1,T_1^{\#}y_2)$ and $h(y_1,T_2x_2) = g(yT_2^{\#},x_2)$

for all $x_1 \in X_1$, $x_2 \in X_2$, $y_1 \in Y_1$ and $y_2 \in Y_2$. Write $\mathcal{L}(\mathcal{P},Q)$ to denote the set of all continuous (T_1,T_2). Then $\mathcal{L}(\mathcal{P},Q)$ for the triple product defined by

$< (R_1,R_2)\,(S_1,S_2)\,(T_1,T_2) > = (R_1S_2^{\#}\,T_1\,,\,T_2S_1^{\#}\,R_2)$

is a ternary algebra. Moreover, if $\mathcal{F}(\mathcal{P},Q)$ denote the set of all $(T_1,T_2) \in \mathcal{L}(\mathcal{P},Q)$ such that both T_1 and T_2 have finite rank then *every ternary subalgebra* A *of* $\mathcal{L}(\mathcal{P},Q)$ *containing* $\mathcal{F}(\mathcal{P},Q)$ *is a prime ternary algebra with socle* $\mathcal{F}(\mathcal{P},Q)$.

2) Let D be a division associative algebra with involution and suppose that (X,g) and (Y,h) are two left vector spaces with nondegenerate inner products over D (both either hermitian or alternate). We recall that in the alternate case D is a field and its involution is the identity. An operator $T \in \hom_D(X,Y)$ is said to be *continuous* if there is $T^{\#} \in \hom_D(Y,X)$ (necessarily unique) such that

$h(xT,y) = g(x,yT^{\#})$

for all $x \in X$, $y \in Y$. We denote by $\mathcal{L}(X,Y)$ the set of all continuous linear operators of (X,g) into (Y,h). It is not difficult to see that the triple product given by

$< T\,S\,R > = TS^{\#}R$

for all $T,S,R \in \mathcal{L}(X,Y)$ defines a structure of ternary algebra on $\mathcal{L}(X,Y)$. Let $\mathcal{F}(X,Y)$ denote the set of all finite rank continuous linear operators of (X,g) into (Y,h). Then *every ternary subalgebra* A *of* $\mathcal{L}(X,Y)$ *containing* F(X,Y) *is prime with socle* $\mathcal{F}(X,Y)$.

We note that ternary algebras A of type (2) are *strongly prime* in the sense that $\langle a A b \rangle = 0$ implies $a = 0$ or $b = 0$ while ternary algebras of type (1) do not enjoy this property. Moreover, in type (2) we can still distinguish between two classes: Ternary algebras A associated with hermitian spaces contain minimal inner ideals I such that $\langle I I I \rangle \neq 0$ while those associated with alternate products do not contain such an ideal I.

Since every semiprime ternary algebra A satisfying dcc on right ideals has socle essential, Theorems 1 and 2 can be applied to determine the structure of such ternary algebras, so obtaining in particular the theorem of Loos for simple ternary algebras satisfying dcc on inner ideals, already cited.

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