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NUMERICAL RADIUS ATTAINING OPERATORS

by

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Paralleling the investigations by J. Lindenstrauss on norm attaining operators, B. Sims raised the question of the norm denseness of numerical radius attaining operators. Up to date, only partial answers to this question have been given. Berg and Sims [2] got an affirmative answer for uniformly convex spaces. The same result was shown by C. S. Cardassi [4] for uniformly smooth spaces. The same author has also shown that numerical radius attaining operators on c_0 , l_1 , $C(K)$ and $L_1(\mu)$ are dense [5,6,7].

In [1] the authors have obtained an isomorphic sufficient condition for the denseness of numerical radius attaining operators, namely reflexivity of the space. In fact, it is shown that, given an operator T on a Banach space X , it can be found a compact operator A , with arbitrarily small norm, such that the second transpose of $T + A$ attains its numerical radius. This result is analogous to the one obtained by Lindenstrauss for norm attaining operators [8; Theorem 1].

Next we give the indispensable definitions and the precise statement of our result.

The dual space of a normed space X will be denoted by X^* , $BL(X)$ will be the normed space of bounded linear operators on X .

DEFINITION. The numerical radius of an operator T in $BL(X)$ is defined by

$$v(T) = Sup \{ \ |f(T(x))| \ : \ (x,f) \in \Pi(X) \ \},$$

where $\Pi(X) = \{ \ (x,f) \in X \times X^* : f(x) = \|f\| = \|x\| = 1\}$ and we say that T attains its numerical radius when there exists (x_0,f_0) in $\Pi(X)$ such that

$$|f_0(T(x_0))| = v(T)$$

We denote by $NRA(X)$ the set of operators on X that attain their numerical radius.

Let T^* denote the transpose operator of T. It is well known that

$$v(T^*) = v(T) \ , \quad [3; \text{Corollary } 9.6].$$

Let us write $NRA_0(X) = \{ T \in BL(X) : T^{**} \in NRA(X^{**})\}$. It is clear that $NRA(X) \subset NRA_0(X)$ and both sets are equal if X is reflexive.

The main result in [1] is the following:

THEOREM. For every Banach space X , $NRA_0(X)$ is norm-dense in $BL(X)$. Moreover, every operator on X may be perturbed by a compact one, of arbitrarily small norm, to obtain an operator in $NRA_0(X)$.

COROLLARY. If X is a reflexive Banach space, then $NRA(X)$ is norm dense in $BL(X)$.

For the proof of the above theorem we use the following characterization of operators in $NRA_0(X)$ which may be of interest in itself.

LEMMA. Let X be a normed space and $S \in BL(X)$. Assume that there are sequences $\{x_n\}$ in X , $\{f_n\}$ in X^* and $\{\delta_n\}$, $\{\varepsilon_n\}$ in $\mathbb{R}^+$, satisfying

(a) $$\| f_n \| = \| x_n \| = 1$$

(b) $$\{\delta_n\} \to 0, \ \{\varepsilon_n/\delta_n\} \to 0 \ \text{ and}$$

(c) $$1 + \delta_n v(S) \leq |f_n(x_{n+k})| + \delta_n |f_n(S(x_{n+k}))| + \varepsilon_n$$

for all positive integers n and k. Then $S \in NRA_0(X)$.

Conversely, if $S \in NRA_0(X)$ and $\{\delta_n\}$, $\{\varepsilon_n\}$ are sequences of positive real numbers satisfying the condition (b) of the Lemma, then sequences $\{x_n\}$ and $\{f_n\}$ can be chosen so as to satisfy (a) and (c).

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