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LOCAL RINGS WHOSE FORMAL FIBERS ARE COMPLETE INTERSECTIONS

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All rings considered in this note will be commutative and noetherian. A^* will denote the completion of the local ring A.

M. André |1, supplément, cor.31| has established the following result: a local ring A is quasi-excellent if and only if $H_1(A,A\ ,-) = 0$.

Here $H_n(A,B,-)$ denotes the n-th André-Quillen homology functor of the A-algebra B.

The aim of this note is to state an analogous result in "dimension 2", that is to say:

<u>Theorem P</u>. Let A be a local ring whose formal fibers are complete intersections (c.i.). Then

1) for every $p \in \mathrm{Spec}(A)$, the formal fibers of A_p are c.i.

2) whenever B is a finitely generated A-algebra, the set

$$CI(B) := \{ p \in \mathrm{Spec}(B) / B_p \text{ is c.i.} \}$$

is open in Spec(B).

Observe that properties 1) and 2) are obtained by changing the word "regular" to "c.i." throughout the definition of quasi-excellent ring. On the other hand, the formal fibers of A are c.i. if and only if the natural homomorphism $A \longrightarrow A^*$ is c.i. and this is equivalent to the vanishing of $H_2(A,A^*,-)$.

We recall that a local ring A is c.i. if A^* is isomorphic to R/J, where R is a regular local ring and J is generated by a regular sequence. A noetherian ring A is c.i. if A_p is a local c.i. for each $p \in \mathrm{Spec}(A)$.

In the proofs we use two results of L.L. Avramov |3|:

— If A is a local c.i., then A_p is c.i. for each $p \in Spec(A)$.

— If $\phi:A \longrightarrow B$ is a flat local homomorphism and K the residue field of A, then B is c.i. if and only if A and $B \otimes_A K$ are c.i.

We start with some results about c.i. homomorphisms.

A homomorphism $\phi:A \longrightarrow B$ is said to be c.i. |5| if it is flat and if, for each $q \in Spec(B)$, with $p = \phi^{-1}(q) \in Spec(A)$ and $K(p) = A_p/pA_p$, the ring $B \otimes_A K(p)$ is c.i.

Proposition 1. Let A be a local c.i. ring. Then the natural homomorphism $A \longrightarrow A^*$ is c.i.

Theorem 1. Let $\phi:A \longrightarrow B$ be a flat homomorphism. The following properties are equivalent:

1) ϕ is c.i.

2) $H_2(A,B,-) = 0$

3) $H_3(A,B,-) = 0$

4) $H_n(A,B,-) = 0$ for n sufficiently large.

The equivalence between 2) and 3) follows from a result of Gulliksen |1, th. 17.13|, and the equivalence between 2) and 4) is a consequence of a theorem of Avramov |3, th. 1|.

Using this characterization and the properties of the André-Quillen homology |1|, we prove the following two propositions.

Proposition 2. Let $\phi:A \longrightarrow B$ and $\psi:B \longrightarrow C$ be two homomorphisms.

1) If ϕ and ψ are c.i., then $\psi\phi$ is c.i.

2) If $\psi\phi$ is c.i. and ψ is faithfully flat, then ϕ is c.i.

3) If ϕ and ψ are flat and $\psi\phi$ is c.i., then ψ is c.i.

Proposition 3. Let $\phi:A \longrightarrow B$ be a c.i. homomorphism and C a finitely generated A-algebra. Then the induced homomorphism $\psi:C \longrightarrow B \otimes_A C$ is c.i.

We can now obtain the results of which theorem P is a corollary.

Theorem 4. Let $\phi: A \longrightarrow B$ be a faithfully flat c.i. homomorphism. If CI(B) is open in Spec(B), then CI(A) is open in Spec(A).

Theorem 6. Let A be a ring such that the formal fibers of $A_{\mathfrak{m}}$ are c.i. for each maximal ideal $\mathfrak{m}$ of A. Then the formal fibers of $A_{\mathfrak{p}}$ are c.i. for each $\mathfrak{p} \in \mathrm{Spec}(A)$.

Proof of theorem P. 1) It is a consequence of theorem 5.

2) Since the natural homomorphism $\phi: A \longrightarrow A^*$ is c.i., proposition 3 shows that the induced homomorphism $\psi: B \longrightarrow A^* \otimes_A B$ is c.i. Moreover ψ is faithfully flat. By Cohen's theorem |1, th. 16.30|, A^* is homomorphic image of a regular local ring, and so the finitely generated A^*-algebra $A^* \otimes_A B$ is a homomorphic image of a regular ring. Hence |4, cor. 3.4| the set $CI(A^* \otimes_A B)$ is open in $\mathrm{Spec}(A^* \otimes_A B)$. Application of theorem 4 yields that CI(B) is open in Spec(B).

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