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FACTORIZATION OF OPERATORS AND DUALITY

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The well-known factorization of Davis-Figiel-Johnson-Pelczynski [6], briefly DFJP factorization, shows how an operator in the class L of all bounded linear operators between Banach spaces, $T \in L(E,X)$, factors through a Banach space Y in such a way that in the corresponding product $T = jA$ the operator j is tauberian injective; moreover, Y is reflexive if and only if T is weakly compact ($T \in WCo$). This construction has been used by many authors and systematically studied in [22] and [23].

We denote $H(E) := E''/E$ and $H(T) \in L(H(E),H(X))$ is the operator induced by the biconjugate T" of T in the form $H(T)(x''+E) = T''x'' + X$. T is said to be tauberian provided $(T'')^{-1}(X) \subset E$; of course $T''(E) \subset X$, so actually T is tauberian if and only if $(T'')^{-1}(X) = E$, [14], [29], or equivalently H(T) injective. Then, T is said to be a cotauberian operator provided H(T) has range dense in H(X); note that our cotauberian operators are different from those considered by K.W.Yang in [30]. It is clear that T is cotauberian if and only if T' is tauberian.

With every operator ideal $\mathfrak{U}$ there is associated the Banach space ideal $Sp(\mathfrak{U}) := \{E \in B \mid I_E \in \mathfrak{U}\}$, and with every Banach space ideal A there is associated the operator ideal $Op(A) := \{T \in L \mid T$ factors through some $E \in A\}$; then $A = Sp(Op(A))$, but $Op(Sp(\mathfrak{U})) \subset \mathfrak{U}$. The three-space property in the framework of a space ideal A was considered in [25] and means that if $M \subset X \in B$ and $M, X/M \in A$, then $X \in A$. The factorization property for an operator ideal $\mathfrak{U}$ means $\mathfrak{U} = Op(Sp(\mathfrak{U}))$.

The paper is organized as follows.

In Section 1 we show a factorization $T = Uk$ of $T \in L(E,X)$ through a Banach space Z, where k is a cotauberian operator with dense range, and investigate relations between both factorizations:

1.1. <u>PROPOSITION</u> Let $T \in L(E,X)$ and $T = Uk$ its cotauberian factorization. Then: (i) k is cotauberian with range dense. (ii) $T' = k'U'$ is the tauberian factorization of T', and the intermediate space Y is dual of that Z of the cotauberian factorization of T.

The remainder of the section contains more or less immediate applications of these factorization techniques.

In section 2 we consider functions S from the class B of all Banach spaces in the class N of all normed spaces, named ideal functions, which assign to every $E \in B$ a linear subspace $S(E)$ of E'' such that $E \subset S(E)$ and $T''(S(E)) \subset S(X)$ for every $E, X \in B$ and $T \in L(E,X)$. Each ideal function S determine two operator ideals $\mathcal{U}^S$ and $\mathcal{U}_S$ defined by

$$\mathcal{U}^S(E,X) := \{T \in L(E,X) \mid T''(S(E)) \subset X\} \text{ and } \mathcal{U}_S(E,X) := \{T \in L(E,X) \mid T''E'' \subset S(X)\}.$$

We shall say that S is injective if for every $E \in B$ and every subspace M of E we have $i''(S(M)) = S(E) \cap M^{oo}$ (i the embedding map); and S is surjective if for every $E \in B$ and every subspace M of E we have $q''(S(E)) = S(E/M)$ (q the quotient map).

The main results in section 2 are the following:

2.6. <u>THEOREM</u>. (i) If S is injective, then $Sp(\mathcal{U}^S)$ is three-space.

(ii) If S is surjective, then $Sp(\mathcal{U}_S)$ is three-space.

2.10. <u>THEOREM</u> Let S be a closed ideal function, that is, such that $S(F) \in B$ for every Banach space F; let $X, E \in B$, W a bounded absolutely convex of X, and p a continuous seminorm in E.

(i) If S is injective and $W^{oo} \subset S(X)$, then the intermediate space Y of the tauberian (DFJP) construction belongs to $Sp(\mathcal{U}_S)$, that is, $S(Y) = Y''$.

(ii) If S is surjective and $\{x \in E \mid p(x) \leq 1\}^{o}$ is relatively $\sigma(E', S(E))$-compact, then the intermediate space Z of the cotauberian construction belongs to $Sp(\mathcal{U}^S)$, that is, $S(Z) = Z$.

2.11. <u>THEOREM</u> Let S be a closed ideal function.

(i) If S is injective, then $\mathcal{U}_S$ has the factorization property, that is, $\mathcal{U}_S = Op(Sp(\mathcal{U}_S))$.

(ii) If S is surjective, then $\mathcal{U}^S$ has the factorization property, that is, $\mathcal{U}^S = Op(Sp(\mathcal{U}^S))$.

Section 3 contains some examples and remarks.

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