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A CHARACTERIZATION OF SMOOTH AND REGULAR ALGEBRAS IN CHARACTERISTIC ZERO

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All rings considered in this paper will be commutative and with unit. For an A-module M, $fd_A(M)$ (resp. $pd_A(M)$) will denotes the flat (resp. projective) dimension of M. We will use the André-Quillen homology functors $H_n(A,B,-)$ |1|.

Let A be a ring and B an A-algebra. Consider B as a $B\otimes_A B$-algebra via the homomorphism $\pi : B\otimes_A B \longrightarrow B$, $\pi(b\otimes b') = bb'$. The purpose of this paper is to study the condition $fd_{B\otimes_A B}(B) < \infty$. We obtain:

Main Theorem. Let A be a noetherian ring and B an A-algebra. Suppose that the characteristic of B is zero and that $B\otimes_A B$ is a noetherian ring. The following statements are equivalent:

1) The homomorphism $A \longrightarrow B$ is regular (i.e. $H_1(A,B,-) = 0$)

2) B is a flat A-module and $fd_{B\otimes_A B}(B) < \infty$.

Corollary A. Let A be a noetherian ring and B an A-algebra of finite type. Suppose that the characteristic of B is zero. Then, B is a smooth A-algebra if and only if B is a flat A-module and $fd_{B\otimes_A B}(B) < \infty$.

Corollary B. Let K be a zero characteristic field and B a K-algebra of finite type. Then, B is a regular ring if and only if $fd_{B\otimes_A B}(B) < \infty$.

Corollary C. Let K be a zero characteristic field and B a local K-algebra of essentially finite type. Then, B is a regular local ring if and only if $fd_{B\otimes_A B}(B) < \infty$.

In order to prove Main Theorem we use the following results.

Proposition 1. Let A be a noetherian ring and B an A-algebra of finite type. Then, $H_2(A,B,-) = 0$ if and only if $fd_A(B) < \infty$ and $H_3(A,B,-) = 0$.

It is a consequence of |4, corl. 3.2.2|, |3, th. 1'| and |1, prop. 17.2|.

Proposition 2. Let A be a noetherian ring and B a flat and noetherian A-algebra. The following conditions are equivalent:

i) $H_2(A,B,-) = 0$

ii) $H_n(A,B,-) = 0$ for n sufficiently large.

In the proof of ii) $\Rightarrow$ i) we use a deep Avramov's result |2, th. 1|.

Proposition 3. Let A be a ring, B a flat A-algebra, and W a B-module. For each $n \geq 0$ there is an isomorphism

$$H_n(A,B,W) \simeq H_n(B\otimes_A B,B,W).$$

From proposition 1 and 3 it follows:

Theorem 4. Let A be a ring and B a flat A-algebra. Suppose that $B\otimes_A B$ is a noetherian ring. Then, $H_1(A,B,-) = 0$ if and only if $fd_{B\otimes_A B}(B) < \infty$ and $H_2(A,B,-) = 0$.

Proof of Main Theorem

1) $\Rightarrow$ 2). It is a consequence of |1, corol. 15.20| and theorem 4.

2) $\Rightarrow$ 1). Since $B\otimes_A B$ is a noetherian ring and B is a $B\otimes_A B$-module finitely generated, we have $fd_{B\otimes_A B}(B)<\infty$ if and only if $pd_{B\otimes_A B}(B) < \infty$. Therefore |5, th. 8.6| $H_r(A,B,B) = 0$ for r sufficient large and $H^s(A,B,-) = 0$ for s sufficient large. This implies $H_n(A,B,-) = 0$ for n sufficient large. Using proposition 2 and theorem 4, the result follows.

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