EXTRACTA MATHEMATICAE 2, n.3, 102-104 (1987)

FINITELY ADDITIVE INTEGRATION

M. Díaz Carrillo, P. Bobillo Guerrero

Dpto. de Análisis Matemático. Fac. de Ciencias
Univ. de Granada. 18071-Granada

AMS class.: 28A10, 28A20, 28A25.

1. In [1] we generalize the process of prolongation of a Daniell-Bourbaki integral. We consider a vector lattice B of real-valued functions on a set $X \neq \Phi$, under the pointwise defined operations and relations $+, t\cdot, =, \leq, \vee, \wedge$. The treatment presented here differs in that we are concerned with a nonnegative linear functional I on B, which in general is not continuous in any sense. The triple (X,B,I) is called a *Loomis system*.

The initial elemental integral I is extended to a class $L(B,I)$ of I-summable functions by the previous introduction of the appropriate oscillation integrals, and then we consider the I-integrable functions, which form the widest possible class of those to which the functional I may be prolonged. The obtained results have an outstanding paralelism with the well known Daniell process. Also, in [2] we study the measurability with respect to a Loomis system.

2. We assume given (X,Ω,μ) where Ω is a ring of subsets of X and μ a nonnegative finitely additive measure on Ω.

A subset C of X is called μ-*null* if for each $\varepsilon \in R^+$, there is $A_\varepsilon \in \Omega$ with $C \subset A_\varepsilon$ and $\mu(A_\varepsilon) < \varepsilon$. Ω_0 denotes the class of all μ-null sets. Now, we define $\bar{\Omega} = \{(A-A_1) \cup A_2;\ A \in \Omega,\ A_1, A_2 \in \Omega_0\}$. $\bar{\Omega}$ is a ring, and μ can be extended to $\bar{\Omega}$ by the formula $\mu((A-A_1)\cup A_2) = \mu(A)$; the definition given to μ is independent of the particular decomposition which is used, and μ is a finitely additive measure on $\bar{\Omega}$.

A function $f \in R^X$ is called *simple* if it can be expresed as $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $A_i \in \bar{\Omega}$, $i = 1,\dots,n$, mutually disjoint, and where $a_i \in R$. The function f will called I_μ-*simple* function whenever $a_i \neq 0$ implies $\mu(A_i) < +\infty$, $i = 1,\dots,n$. Similarly, in the case where $A_i \in \Omega$ we consider the μ-*simple* function (which coincides with the definition given by Dunford-Schwartz, [4]). $S = S(X,\Omega,\mu)$ denotes the class of all μ-*sim-*

ple functions, and $B = B(X,\bar{\Omega},\mu)$ the class of all I_μ-*simple* functions. Clearly $S \subset B$. If f is a simple function, then the finite real number $\sum_{i=1}^{n} a_i\mu(A_i)$ is called the integral of f, and is denoted by $I_\mu(f)$ if $f \in B$, and $\int f\, d\mu$ if $f \in S$. Then, the triple (X,B,I_μ) is a bounded stonian Loomis system, hence we can be consider our extension of [1] or completion of B with respect to I_μ. Thus, $f \in \bar{R}^X$ is said to be I_μ-*summable* function if $f \in L_\mu := L(B,I_\mu) = B_0$ in [1] .

We study now the relationship between the I_μ-summable and μ-integrable functions, which latter, for real valued functions, were presented essentially to Loomis [7] , for Banach space-valued functions, were been introduced to Dunford-Schwartz [4] , and bit more generally to Günzler [5],[6].

3. The main results obtained can be summarized as follows.

a. The class of all the *Riemann-integrable functions* of [6] and [7], i.e. $R^1_e(\mu) := \{f \in R^X;$ to each $\varepsilon > 0$, there are $h,k \in S$ with $h \leq f \leq k$ and $\int (k-h)\, d\mu < \varepsilon\}$, is contained in $L(B, I_\mu)$.

b. If Ω is an algebra (then $\bar{\Omega}$ is so), every μ-*integrable function* in accordance with Dunford-Schwartz definition. i.e. $f \in L(\mu)$ iff there exists a sequence $\{g_n\}_{n \in N}$ of μ-simple function which is a μ-Cauchy sequence and with $g_n \to f$ in μ-measure, then $f \in L(B,I_\mu)$ and $I_\mu(f) = \int f\, d\mu$.

c. In [3] we find that if f is an *abstract-Riemann-integrable function* , i.e., $f \in R^1(\mu) := \{f \in R^X;$ for each $g \in S$, $g \wedge f \in R^1_e(\mu)$ and the set $\{\int g\, d\mu;\ g \in S,\ 0 \leq g \leq f\}$ is bounded$\}$,(see [6]), then f may be expressed as the sum of a I_μ-summable function and a μ-null function.

Note that for any Banach space K, any algebra Ω from X and any finitely additive measure $\mu:\Omega \to [0, +\infty[$, $R^1(\mu)$of [6] coincides with the class $L(\mu)$ of all μ-integrable functions of [4].

d. The result c. can be shaspened as follows (Günzler, private conmunication): If Ω is a semiring from X , $\mu:\Omega \to [0 ,+\infty[$ is finitely additive, and $U \subset X$ with $U = A_1 \cup A_2 \cup \ldots$ with $A_n \in \Omega$ and $\chi_{A_1 \cup \ldots \cup A_n} \to \chi_U$ μ-locally as $n \to +\infty$ (see [5]), then for any $f \in R^1(\mu)$ with $f = 0$ outside U one has $f \in L(B, I_\mu)$.

For σ-additive μ , $U = X$ is possible if $X = A_1 \cup A_2 \cup \ldots$ with $A_n \in \Omega$, p.e., $X = R^n$, Ω ={halfopen $I_1 \times \ldots \times I_n$; I_k of form $[a,b[\subset R\}$, $\mu = \mu^n_L$ = Lebesgue-measure ; so $R^1(\mu^n_L/\Omega , R) \subset L(S, \int .\, d\mu^n_L)$.

Since for any μ always $X = U$ is possible if $X \in \Omega$, with

$L(\mu) = R^1(\mu, R)$, one gets as special case: any μ-integrable function of [4] belong to $L(B, I_\mu)$, for arbitrary μ.

REFERENCES

1. BOBILLO GUERRERO, P. and M. DIAZ CARRILLO. Summable and integrable functions with respect to any Loomis system. To appear in *Arch. Math.*, (1987).
2. ------ Fonctions fortement-mesurables et mesurables par rapport à un système de Loomis. *Bull. Soc Roy. Sci. Liège, 55,4, pp. 467-71, (1986).*
3. ------ On the summability of certain μ-integrable functions. Preprint Univ. de Granada, (1987).
4. DUNFORD N. and J.T. SCHWARTZ. Linear operators, part I, New York,1963.
5. GÜNZLER, H.. Linear functional which are integrals. *Rend. Sem. Mat. Fis. Milano, XLIII, pp. 167-76, (1974)*
6. ------ Integration. Bibliographisches Institut. Wissenschaftsverlag. Zürich. 1985.
7. LOOMIS, L.H. Linear functional and content. *Amer. J. Math.,76, pp.168-182, (1954).*