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"CONJUGACY CLASSES IN FINITE GROUPS II" (*)

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In the following G is a finite group. For each natural number $n = p_1^{a_1} \dots p_t^{a_t}$, with p_i prime and $p_i \neq p_j$ for every $i \neq j$, we define the number, $d_i = d_i(n)$, to be the greatest divisor of $p_i - 1$ which is coprime to n, and $\delta_i = \delta_i(n)$ by

$$\delta_i = \delta_i(n) = \begin{cases} \text{g.c.d.}(p_j^2 - 1 \mid 1 \le j \le t,\ j \neq i) & \text{if } a_i = 1 \\ \text{g.c.d.}(p_j^2 - 1 \mid 1 \le j \le t\ ) & \text{if } a_i > 1, \end{cases}$$

where g.c.d.$(m_i \mid i \in I)$ denotes the greatest common divisor of the family of numbers $(m_i \mid i \in I)$ and we write $\delta_1 = 1$, in case $t = a_1 = 1$. In addition, we use the notation $D(n) = \text{g.c.d.}(d_1\delta_1, \dots, d_t\delta_t)$.

Let r(G) be the number of conjugacy classes of elements of G. In [1], G. Amit and D. Chillag prove the following congruence for finite groups of odd orders:

$$r(G) \equiv |G| \ (\text{mod. } D(|G|)),$$

by using character theory. The above congruence improves (when $|G|$ has some primary power of exponent 1) one well-Known A. Mann's congruence (cf. [2]):

$$r(G) \equiv |G| \ (\text{mod. } d(|G|).\delta(|G|)),$$

where $\delta(n) = \text{g.c.d.}(p^2 - 1 \mid p \text{ is a prime dividing } n)$ and $d(n) = \text{g.c.d.}(p - 1 \mid p \text{ is a prime dividing } n)$.

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We now consider the following numbers

$$\mu_{p_i} = \mu_{p_i}(n) = \text{g.c.d.}(p_j-1 \mid 1\le j\le t,\ j\ne i)$$

$$\sigma_{p_i} = \sigma_{p_i}(n) = (2^{e(p_i)}.d(n).\mu_{p_i}(n))/\text{g.c.d.}(2^{e(p_i)}.d(n).\mu_{p_i}(n)\ ;\ n)$$

where $e(p_i) = 0$ if $p_i=2$, and for $p_i\ne 2$, $e(p_i) = 1$ or 0 according as $\mu_{p_i}(n)/d(n)$ is even or odd, respectively. In addition, for each natural number s, and fixed the arrangement of the primary power dividing n, we define

$$\delta_i^{(s)} = \delta_i^{(s)}(n) = \begin{cases} \text{g.c.d.}(p_j^2-1 \mid 1\le j\le t,\ j\ne i) & \text{if } 1\le i\le s \\ \text{g.c.d.}(p_j^2-1 \mid 1\le j\le t\ ) & \text{if } i \ge s+1. \end{cases}$$

In this work, the following congruence is proved without using character theory:

$$r(G) \equiv |G| \ (\text{mod. l.c.m.}(\ \delta(|G|)d(|G|);\ \sigma_p(|G|),\ p\in\Gamma)),$$

where

$\Gamma = \{\ p \mid p$ is a prime dividing $|G|$, and G has abelian Sylow p-subgroups $\}$.

The above congruence improves G.Amit-D.Chillag's congruence. Indeed, if

$$\Gamma = \{p_1,\ldots,p_u\}\ ,$$

then the above congruence yields

$$r(G) \equiv |G| \ (\text{mod. } D_{(u)}(|G|)/\text{g.c.d.}(D_{(u)}(|G|),|G|)),$$

where

$$D_{(s)}(n) = \text{g.c.d.}((p_1-1)\delta_1^{(s)}(n),\ldots\ldots,(p_t-1)\delta_t^{(s)}(n)).$$

Clearly, $D(|G|)$ divides $D_{(u)}(|G|)/\text{g.c.d.}(D_{(u)}(|G|),|G|)$ and in general are distinct numbers, since Γ contains all prime numbers dividing $|G|$ and with exponent less than 3. Further, the number

$$D_{(u)}(|G|)/\text{g.c.d.}(D_{(u)}(|G|),|G|)$$

may have common factors with $|G|$, whereas $D(|G|)$ and $|G|$ are relatively prime numbers. Two examples at the end show that our results are in some cases best possible.

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[3] Antonio Vera-López "Conjugacy classes in finite groups" Proc. Roy. Soc. Edinburgh Sect. A 105 (1987), 259-264.

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