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ORTHOGONALLY COMPLETE ALTERNATIVE RINGS

by

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Order relations on different sorts of rings have been investigated by a number of people; on associative rings, for example. by Abian [1] and Chacron [5] ; on alternative rings by Myung and Jimenez [9] , and recently on Jordan rings by González and Martínez [6] y [7]. In the case of alternative (and associative) rings , the usual order on Boolean rings ($a \leq b$ iff $ab = a^2$) is also an order and has been used to characterize those reduced rings (no nonzero nilpotent elements) which are a direct product of division rings. One of the conditions used in this characterization is the notion of orthogonal completeness. A subset X of a reduced ring A is called orthogonal if $ab = 0$, for all a,b in X with $a \neq b$. A is said to be orthogonally complete if every orthogonal subset of A has a supremum in A. Haines [8] introduces a generalization of orthogonality which he calls quasi-orthogonality and Burgess and Raphael [3] call boundable. A subset X of a ring A is called boundable if for all a.b in X , $ab(a-b) = 0$, and A is said to be complete if every boundable subset of A has a supremum in A.

Burgess and Raphael [3] prove that a von Neumann regular associative ring is orthogonally complete if and only if it is complete. The main of this paper is to relate the notions of orthogonality and boundability for reduced alternative rings and extend to the von Neumann regular alternative rings the above mentioned result of Burges and Raphael.

In what follows A always stans for a reduced alternative ring The rlation over A given by: $a \leq b$ iff $ab = a^2$ is a partial order on A .

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It is known that a reduced alternative but commutative ring is associative and there exist reduced alternative rings wich are not associative (see [2] or [9]). Other known properties of A are the following:

1) $x \leq y$ implies that $xy = yx$ and $(x,y,z) = (xy)z - x(yz) = 0$ for every z in A.
2) $xy = 0$ if and only if $yx = 0$.
3) $(xy)z = 0$ if and only if $x(yz) = 0$.
4) $xy^2 = 0$ if and only if $xy = 0$
5) The idempotent elements of A commute with every element of A.

Obviously if the ring A is complete, then it is orthogonally complete. There exist reduced alternative but nonassociative rings which are orthogonally complete but non complete. It is clear that every orthogonal subset X of A is a boundable subset and any set which has a supremum with respect $\leq$ is boundable

An alternative ring A is von Neumann regular if for any element a in A there exists an element b in A such that a = aba.

The following results are proved:

Lemma. Let A be an alternative regular ring and let x be an element of A. Then there exists an element $\bar{x}$ in A such that:

i) $x = x\bar{x}x$ and $\bar{x} = \bar{x}x\bar{x}$

ii) $x\bar{x} = \bar{x}x$ is an idempotent (so central) element of A

iii) $\bar{x}$ is the unique element of A with the two above conditions.

Lemma. Let A be a regular alternative ring and let X be a commutative subset. Then there exists an associative commutative subring B of A with $X \subseteq B$ and B is maximal as regular commutative associative subring of A containing X.

This ring B contains every element x of A satisfying $xb = bx$ for every b in B. In particular B contains every idempotent element of A, and the above mentioned result of [3] assures that this ring B is complete if and only if it is orthogonally complete.

Using the above results, the main theorem can be proved.

THEOREM Let A be a von Neumann regular alternative ring. Then A is complete if and only if it is orthogonally complete.

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