EXTRACTA MATHEMATICAE 2, n.3, 114-116 (1987)

FUNCTIONAL CHARACTERIZATION OF BEST AND GOOD APPROXIMATION IN NORMED PRODUCT SPACES

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AMS Class.(1980): 41A65

ABSTRACT

In |6| C. Dierieck deals with a little but important collection of norms in the product of a finite number of normed linear spaces and he extends to such products some results on functional characterization of best approximations.

In this paper we stablish the widest scope in which the mentioned results remain valid.

Let E_1 and E_2 be linear spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) endowed with norms $\| \|_1$ and $\| \|_2$. We shall say that a norm $\| \|$ in $E_1 x E_2$ is of <u>type M</u> when

$$\|x_1\|_1 \leq \|y_1\|_1, \quad \|x_2\|_2 \leq \|y_2\|_2 \Longrightarrow \|(x_1,x_2)\| \leq \|(y_1,y_2)\|$$

The importance in Approximation Theory of the M-norms is due to the following facts:

A norm in $E_1 \times E_2$ is of type M if and only if it verifies any of the following properties:

i) |2| For every $x_k \in E_k$ and every L_k linear subspace (or simply subset) of E_k, $(k=1,2)$

$$y_1 \underset{L_1}{\overset{x_1}{<}} z_1 \quad , \quad y_2 \underset{L_2}{\overset{x_2}{<}} z_2 \Longrightarrow (y_1,y_2) \underset{L_1 \times L_2}{\overset{(x_1,x_2)}{<}} (z_1,z_2)$$

where $y \underset{L}{\overset{x}{<}} z$ means that y is better approximation of x relative to L than z.

ii) For every $x_k \in E_k$ and every L_k subset of E_k, $(k=1,2)$

$$y_1 \in P_{L_1}(x_1) \quad , \quad y_2 \in P_{L_2}(x_2) \Longrightarrow (y_1,y_2) \in P_{L_1 \times L_2}(x_1,x_2)$$

where $y \in P_L(x)$ means that y is best approximation of x relative to L.

iii) |3| The above property (ii), but for L_k linear subspace of E_k, in the case that $\mathbb{K} = \mathbb{R}$ and any of the E_k is of dimension $\geqslant 2$.

On the other hand in the passage to the topological dual of an M-normed linear space we find a kind property:

|1|. If $E_1 \times E_2$ is an M-normed linear space and if the product of topological duals $E'_1 \times E'_2$ is endowed with the M-norm

$$\|(\varphi_1,\varphi_2)\| = \sup\left\{\|\varphi_1\|_1\|x_1\|_1 + \|\varphi_2\|_2\|x_2\|_2 : \|(x_1,x_2)\| = 1\right\}$$

then the mapping $T: E'_1 \times E'_2 \longrightarrow (E_1 \times E_2)'$ defined by

$$T(\varphi_1,\varphi_2)(x_1,x_2) = \varphi_1(x_1) + \varphi_2(x_2)$$

is an isometric algebraic isomorphism.

With the above results we have the essential tool to stablish peculiar formulations in M-normed product spaces of many well known functional characterizations of best and good approximations. For instance:

If L_k is a linear subspace of E_k and $x_k \in E_k \setminus \bar{L}_k$ then the fact $y_{ok} \in P_{L_k}(x_k)$ is characterized by the existence of $\varphi_k \in E'_k$ such that (see e.g. |8| p.18)

$$\|\varphi_k\|_k = 1 \quad ; \quad \varphi_k(y_k) = 0, (y_k \in L_k) \quad ; \quad \varphi_k(x_k - y_{ok}) = \|x_k - y_{ok}\|_k \ .$$

It is also characterized by the existence, for every $y_k \in E_k$, of $\varphi_{y_k} \in E'_k$ such that (see e.g. |8| p.58)

$$\varphi_{y_k} \in \mathrm{Ext}(B'_k) \quad ; \quad \mathrm{Re}\,\varphi_{y_k}(y_k - y_{ok}) \leq 0 \quad ; \quad \varphi_{y_k}(x_k - y_{ok}) = \|x_k - y_{ok}\|_k$$

where $\mathrm{Ext}(B'_k)$ denotes the set of extremal points of the unit ball of E'_k.

If $E_1 x E_2$ is M-normed then we have that $(y_{o1}, y_{o2}) \in P_{L_1 x L_2}(x_1, x_2)$ and, among others, the following results:

|1|. The fact $(y_{o1},y_{o2}) \in P_{L_1 \times L_2}(x_1,x_2)$ is characterized, in the sense of (|8| p.18), by a continuous linear functional of the form $(r_1\varphi_1, r_2\varphi_2)$, where the φ_k are the above mentioned and the r_k are non negative real numbers.

Also it is characterized in the sense of (|8| p.58) by $(r_{y1}\varphi_{y1}, r_{y2}\varphi_{y2})$, with φ_{y_k} the above mentioned and $r_{y_k} \geq 0$.

As partial reciprocous of this results, also in M-normed spaces, we have:

If $(y_{o1},y_{o2}) \in P_{L_1 \times L_2}(x_1,x_2)$ then either $y_{o1} \in P_{L_1}(x_1)$ or $y_{o2} \in P_{L_2}(x_2)$, and both possibilities if the norm in $E_1 \times E_2$ is of type MS, i.e. such that $\|(u_1,u_2)\| < \|(v_1,v_2)\|$ when any of the signs $\|u_1\|_1 \leq \|v_1\|_1$, $\|u_2\|_2 \leq \|v_2\|_2$ is of strict inequality. Moreover if $(\varphi_1,\varphi_2) \in E'_1 \times E'_2$ characterizes the fact $(y_{o1},y_{o2}) \in P_{L_1 \times L_2}(x_1,x_2)$ and if $\varphi_k \neq 0$, $(k=1,2)$, then $\varphi_k \|\varphi_k\|_k^{-1}$ characte *rizes the fact* $y_{ok} \in P_{L_k}(x_k)$.

Analogously for the case (|8| p.58).

With the same techniques used to stablish the above results it is no dificult to obtain formulations in M-normed product spaces of other known results (4, 5, 6, 7, 8) on functional characterization of best and good approximations relative to linear subspaces or simply subsets.

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(To appear in Bolletino U.M.I.)