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The Sum problem for Hilbert spaces.

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We consider the following problem: there are locally convex spaces E for which the sum space $\bigoplus_N E$ is a subspace of some product E^I, and others for which such an embedding is not possible. Examples of the first kind are E= $C^\infty(R)$ or $[l_\infty / \mu(l_\infty, l_1)]$. Examples of the second kind are the spaces carrying the weak topology.

Problem. Characterize those locally convex spaces E such that $\bigoplus_N E$ is (isomorphic to) a subspace of some product E^I.

In this paper we give a complete solution of this problem for Hilbert spaces:

Theorem. Let H be a Hilbert space. Then the locally convex sum $\bigoplus_I H$ is a subspace of the product H^J if and only if H is infinite-dimensional, I is countable, and card J $\geqslant 2^{\aleph_0}$.

Obviously H needs to be infinite-dimensional since $\bigoplus_N K$ does not carry the weak topology and therefore cannot be a subspace of K^I. It is also clear that card J $\geqslant 2^{\text{card} I}$ since this is the cardinal of a base of neighborhoods of 0 of the sum space.

The proof can be divided into some propositions:

Proposition 1. $\bigoplus_N l_2$ **is a subspace of** l_2^J **when card J** $\geqslant 2^{\aleph_0}$

Proposition 2. $\bigoplus_I l_2$ **is not a subspace of any product of copies of** l_2

Then we only need to consider the possibility of an embedding of $\bigoplus_I l_2(\Lambda)$ into $l_2(\Lambda)^J$ I, Λ, J uncountable.

We have:

Lemma. Let I,J be uncountable sets, $p > q$ **and** $T: l_p(I) \longrightarrow l_q(J)$ **a continuous operator. Then** $\text{Im}T \subset l_q(N)$.

The preceding lemma asserts that such a T can only "move" a countable number of coordinates in J. From this follows:

Lemma. A diagonal operator $D_\sigma : l_p(I) \longrightarrow l_p(I)$, $\sigma_i > 0 \ \forall i \in I$ cannot be continuously factorized through $l_q(I)$ if $p \neq q$.

Proposition 3. Let I be an uncountable set, and H a Hilbert space, then $\bigoplus_I H$ is not a subspace of any product H^J.

The idea is that if $\bigoplus_I l_2(I)$ is a subspace of $l_2(I)^J$ then a subfactorization of the diagonal operator

$$\begin{array}{ccc} D_\sigma : l_1(I) & \xrightarrow{\hspace{2cm}} & l_1(I) \\ & \searrow \quad \nearrow & \\ & l_2(I) & \end{array}$$

($\sigma_i > 0$ for all i) through $l_2(I)$ could be done. Using the orthogonal projection onto $\overline{\text{ImA}}$ we get a factorization

$$\begin{array}{ccc} D_\sigma : l_1(I) & \xrightarrow{\hspace{2cm}} & l_1(I) \\ & \searrow \quad \nearrow & \\ & l_2(I) & \end{array}$$

but then, by the lemma, $\text{ImD}_\sigma \subset l_1(N)$, which is impossible since D_σ "moves" all the indexes.

References

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