EXTRACTA MATHEMATICAE 3, n.2, 49-51 (1988)

ON THE THREE SERIES THEOREM IN NUMBER THEORY

Jesús de la Cal

Departamento de Matemáticas. Universidad del País Vasco.
48080 Bilbao.

AMS 1980 Class.: Primary 10K20; secondary 60B11.

Research supported by the University of the Basque Country.

Let (f_n) be a sequence of arithmetic functions (i.e. functions of natural argument) with values in a real, separable Banach space $(\mathfrak{X}, ||\cdot||)$, and, for each n let P_n be the probability measure on the set N of natural numbers assigning the weight $1/n$ to each $k = 1,\ldots,n$ and 0 otherwise. If the sequence of probability measures on $\mathfrak{X}$, $(P_n f_n^{-1})$ converges weakly to a probability measure μ on $\mathfrak{X}$, we write $f_n \Longrightarrow \mu$, or $f_n \Longrightarrow X$ if μ is the probability distribution of the $\mathfrak{X}$-valued random variable X. When $f_n = f$, all n, we simply write $f \Longrightarrow \mu$, or $f \Longrightarrow X$, and μ is said to be the limit distribution of f.

The following theorem gives sufficient conditions for the existence of the limit distribution for an $\mathfrak{X}$-valued additive function, i.e. a function $f: N \longrightarrow \mathfrak{X}$ such that $f(mn) = f(m)+f(n)$ whenever m and n are relatively prime. This result generalizes the part "if" of the famous Erdös-Wintner theorem for real additive functions (see, for instance |2|, Th.7.1 and 7.2).

THEOREM 1. Suppose $\mathfrak{X}$ r-smoothable $(1 < r \leq 2)$ and let f be an $\mathfrak{X}$-valued additive function. If the two series

$$\sum_{||f(p)|| \geq c} 1/p \quad , \quad \sum_{||f(p)|| < c} \frac{||f(p)||^r}{p}$$

converge for some $c > 0$, and if $\quad a_n = \sum_{||f(p)|| < c,\ p \leq n} \frac{f(p)}{p}$, then $$f - a_n \Longrightarrow \sum_{\|f(p)\| < c} \left[f(p^{b_p}) - \frac{f(p)}{p} \right] + \sum_{\|f(p)\| \geq c} f(p^{b_p}).$$

Therefore, if a_n also converges (in the norm), then $f \Longrightarrow \sum_p f(p^{b_p})$, or, in case f is strongly additive (i.e. $f(p^k) = f(p)$ for $k = 1, 2, \ldots$) $f \Longrightarrow \sum_p f(p)\ d_p$.

In this statement p ranges the set $\mathcal{P}$ of prime numbers, the b_p are independent, integer-valued random variables, having the geometric distribution

$$P(b_p = k) = (1 - 1/p)\ p^{-k} \qquad k = 0, 1, 2, \ldots,$$

and d_p is the indicator function of the set $(b_p \geqslant 1)$.

A similar result, with the obvious modifications, is true if $\mathfrak{X}$ is only supposed to be of type 2. (For the concepts of type r and r-smoothable see |3| and |5|).

The converse of Theorem 1 (i.e. the convergence of the three series is a necessary condition for the existence of limit distribution of f) holds true if $\mathfrak{X}$ is finite-dimensional, but for general r-smoothable Banach spaces it is an open problem.

The proof of Theorem 1 depends upon the following lemma, which provides an inequality of the Turán-Kubilius type.

LEMMA 1. Let $\mathfrak{X}$ be of type r $(1 < r \leqslant 2)$ and let g be an $\mathfrak{X}$-valued, strongly additive function. There is a constant C, independent of g and n, such that

$$E_n \|g - A_n\|^r \leqslant C \sum_{p \leqslant n} \frac{\|g(p)\|^r}{p},$$

where $A_n = \sum_{p \leqslant n} \frac{g(p)}{p}$ and E_n is the P_n-expectation.

Another consequence of this lemma is that if $\mathfrak{X}$ is of type r and (f_n) is an array of $\mathfrak{X}$-valued additive functions, then the law of large numbers $\frac{f_n - A_n}{\psi_n B_n} \longrightarrow 0$ holds for every sequence of real numbers $\psi_n \longrightarrow \infty$ if $A_n = \sum_{p \leq n} \frac{f_n(p)}{p}$, $B_n = \left(\sum_{p \leq n} \frac{\|f_n(p)\|^r}{p} \right)^{1/r}$ and $\sup_n \frac{\|f_n(m)\|}{B_n} < \infty$, m = 1, 2, ... (C.f. |2| for the case $\mathfrak{X} = R$).

As an application of the preceding results we consider the following: Let f be a real additive function and for each $p \in \mathcal{P}$ and $n \in N$ let $\beta_p(n)$ be the exponent of p in the prime factorization of n. Then $f^* = (f(p_1^{\beta_{p_1}}), \ldots, f(p_k^{\beta_{p_k}}), \ldots)$, where $p_1, p_2, \ldots$ is an arrangement of prime numbers, is an additive function with values in the real, separable Hilbert space l_2. If f is integer-valued and $\sum_{f(p) \neq 0} 1/p < \infty$ we have $f^* \longrightarrow X \overset{\text{def}}{=} (f(p_1^{b_{p_1}}), \ldots, f(p_k^{b_{p_k}}), \ldots)$. From this we deduce that the asymptotic density is a probability measure on the σ-field of sets of natural numbers generated by the functions $f(p^{\beta_p})$. Moreover many explicit formulas for densities of sets of natural numbers can be obtained with little effort.

REFERENCES

|1| BILLINGSLEY, P. Convergence of Probability Measures. Wiley, New York, 1968.

|2| BILLINGSLEY, P., The probability theory of additive arithmetic functions. Ann. Probab. 2 (1974), 749-791.

|3| HOFFMANN-JØRGENSEN, J. and PISIER, G. The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. 4 (1976) 587-599.

|4| RUZSA, I.Z. Generalized moments of additive functions. J.Number Theory. 18 (1984) 27-33.

|5| SZULGA, J. Three series theorem for martingales in Banach spaces. Bull. Polish Acad. Sci. Math. 25 (1977), 175-180.

|6| WOYCZYŃSKI, W.A. Asymptotic behavior of martingales in Banach spaces. Springer's Lecture Notes in Math. 526 (1975), 273-284.

To appear in THE ANNALS OF PROBABILITY.