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ENDOMORPHISM RINGS THROUGH EQUIVALENCES [1]

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The use of category equivalences for the study of endomorphism rings stems from Morita's Theorem. In a sense, this theorem can be viewed as stating that if P is a finitely generated projective generator of R-mod and S = End($_R$P), then properties of P correspond to properties of S through the equivalence between the categories R-mod and S-mod given by the functor $Hom_R(P,-)$. Generalizations of this theorem were given, for instance, in [1] and [2]. In [1] P is only assumed to be finitely generated and projective, and $Hom_R(P,-)$ induces in this case an equivalence between S-mod and a quotient category of R-mod, while in [2] it is shown that if P is a finitely generated quasiprojective self-generator then the equivalence provided by the same functor is now defined between the category σ[P] of all the R-modules sub-generated by P and S-mod.

Later on, more general category equivalences were constructed in an analogous way to those already mentioned, by replacing S-mod by a certain quotient category of itself. Thus, in [5] Morita contexts are used to obtain, for an arbitrary module $_R$M, a category equivalence between the quotient categories $_R\mathfrak{U}$ and $_S\mathfrak{U}$ of R-mod and S-mod determined by the two trace ideals of the derived context of M. On the other hand, if M is a Σ-quasiprojective module, then it is shown in [3] that the functor $Hom_R(M,-)$ induces an equivalence between quotient categories $\mathfrak{C}$[M] of σ[M] and (S,$\mathfrak{F}$)-mod of S-mod, and the latter quotient category coincides with S-mod when M is finitely generated.

In any case, two conditions are required in order to obtain, by means of these methods, necessary and sufficient conditions on a module $_R$M (or on a class of R-modules related to M) for its endomorphism ring S to have a specific property: first, a category equivalence between subcategories $\mathfrak{C}_R$ of R-mod and $\mathfrak{C}_S$ of S-mod must exist; second, S must be an object of the category $\mathfrak{C}_S$ so that we can relate properties of S to properties of certain objects of R-mod. If we compare the two constructions just mentioned, we see that while that in [5] is more general, since it applies to any module ${}_R M$, the one in [3] is more effective in that S always belongs to $(S,\mathcal{F})$-mod (but there is no need for S to belong to ${}_S\mathfrak{U}$ in [5]). On the other hand, these two constructions are the same in a sense: each of them considers the torsion theory of R-mod [5] or $\sigma[M]$ [3] in which the torsionfree objects are precisely the M-distinguished objects in the terminology of [4] and the quotient category ${}_R\mathfrak{U}$ or $\mathcal{C}[M]$ corresponds to this torsion theory. So, if we try to unify these two constructions into a more general one (with an arbitrary Grothendieck category $\mathcal{C}$ substituted for either R-mod or $\sigma[M]$), then two questions (corresponding to the above two conditions) arise:

(1) Is there an equivalence between the quotient category of $\mathcal{C}$ with respect to the torsion theory induced by the torsionfree class of M-distinguished objects and a quotient category of S-mod?

(2) Under what conditions do we get that S belongs to the quotient category of S-mod in case the equivalence of (1) does exist?

The answer to (1) turns out to be affirmative under fairly general hypotheses. In fact, for an arbitrary Grothendieck category $\mathcal{C}$, let M be an object of $\mathcal{C}$, **F** the class of M-distinguished objects (in the sense of [4]), (**T**,**F**) the associated torsion theory of $\mathcal{C}$ and $\mathcal{C}_M$ the corresponding quotient category. Then we have:

Theorem 1. If $\mathcal{C}$ is locally finitely generated, then the functor $\mathrm{Hom}_{\mathcal{C}}(M,-):\mathcal{C} \longrightarrow$ S-mod induces an equivalence of categories between $\mathcal{C}_M$ (viewed as a subcategory of $\mathcal{C}$) and $(S,\mathcal{F})$-mod, $\mathcal{F}$ being the left Gabriel topology of S $\{I \subseteq {}_S S \mid M/MI$ is **T**-torsion$\}$.

Concerning question (2), we need to make a couple of definitions in order to explain the answer. First, let us denote by **t** the torsion radical of $\mathcal{C}$ associated to the above torsion theory (**T**,**F**), and by $\bar{M}$ the quotient M/t(M). Then, we say that M is weakly M-distinguished if the following two conditions are verified:

(a) $\mathrm{Hom}_{\mathcal{C}}(M,t(M)) = 0$.

(b) For every morphism $f:M \longrightarrow \bar{M}$ there exists an endomorphism s of M such that $p\circ s = f$, p being the canonical projection $M \longrightarrow \bar{M}$.

On the other hand, we will say that an object X of $\mathcal{C}$ is **T**-M-injective if for each monomorphism $u:L \longrightarrow M$ of $\mathcal{C}$ such that

Coker u is T-torsion, we have that the canonical homomorphism u^*: $\mathrm{Hom}_{\mathscr{C}}(M,X) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(L,X)$ is a surjection. Finally, M will be called weakly T-closed if M is weakly M-distinguished and $\tilde{M}$ is T-M-injective. Then we have:

Theorem 2. In the equivalence of Theorem 1, S is its own ring of quotients if and only if M is weakly T-closed.

Hence this condition is what is needed in order to obtain results about endomorphism rings by using category equivalences. But if $\mathscr{B}$ is a subcategory of a given category $\mathscr{D}$, then an object M may be weakly T-closed in $\mathscr{B}$ while not being so as an object of $\mathscr{D}$. In this sense, we have:

Proposition 3. Let M be a left R-module. If M is weakly T-closed in R-mod, then M is weakly T-closed as an object of $\sigma[M]$ (but not conversely).

This suggests that to study the endomorphism ring of a module M by using the equivalence of categories of Theorem 1, it is preferable to take $\mathscr{C} = \sigma[M]$ than $\mathscr{C}$ = R-mod.

Finally, we state when the equivalence of Theorem 1 results in an equivalence between $\mathscr{C}_M$ and all of S-mod.

Theorem 4. Let $\mathscr{C}$, M, S and $\mathscr{F}$ be as in Theorem 1. Then $\mathscr{F}$ is the trivial filter $\mathscr{F} = \{S\}$ if and only if M is a finitely generated and quasiprojective object of $\mathscr{C}$ which is CQF-3 in the sense of [6].

(An expanded version of this paper will appear in J.Algebra)

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