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Strictly Singular and Strictly Cosingular Operators on C(K,E)

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In this note we present some results on strictly singular and strictly cosingular operators defined on a space of vector valued continuous functions, which will appear in the Mathematische Nachrichten, with the same title.

1 Definitions and General Remarks.

Let E, F be Banach spaces.

1. An operator $T \in \mathcal{L}(E,F)$ is an injection (respect. a surjection) if it is an isomorphism between E and $T(E)$ (respect. it is an onto map).
2. $T \in \mathcal{L}(E,F)$ is a Kato operator, or strictly singular if its restriction to any infinite dimensional subspace of E is not an injection.
3. $T \in \mathcal{L}(E,F)$ is a Pelczynsky operator, or strictly cosingular, if for every surjection $Q \in \mathcal{L}(F,H)$ with H infinite dimensional, $Q \circ T$ is not a surjection.
4. If $T \in \mathcal{L}(E,F)$ is strictly singular, it can not fix a copy of c_0, so by a well known result of Pelczynski ([6]) it is unconditionally convergent. That is also the case for strictly cosingular ...ators when the range is contained in a separably complemented subspace.

...s and terminology we refer the reader to [3], [4] or [5]. Also the interested reader ... for a general information on the subject.

...ESULTS.

...e aim of this paper is the study of strictly singular and strictly cosingular operators on spaces of vector valued continuous functions. Given a compact Hausdorff space K and Banach spaces E, F, any operator (linear continuous map) $T : C(K,E) \longrightarrow F$ has a representing measure m, defined on the Borel σ-field $\beta_0(K)$ of K, in such a way that

$$T(f) = \int f dm$$

(see [3]).

The same formula takes sense when f belongs to $B(\beta_0(K),E)$, the uniform limits of $\beta_0(K)$-simple functions. In this way it defines an extension $\hat{T}$ of T. We shall denote by $|m|$ the semivariation ([3], pg. 51) of m. With this notation, we have:

Theorem 2.1 *An operator $T : C(K,E) \longrightarrow F$ is strictly singular if and only if its extension $\hat{T}$ is strictly singular.*

Corollary 2.2 *Let $T : C(K,E) \longrightarrow F$ be strictly singular with $m \leftrightarrow T$. Then*

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1. $|m|$ *is continuous at* $\emptyset$.
2. *For each* $A \in \beta_0(K)$, $m(A)$ *maps* E *into* F *and is strictly singular.*

For strictly cosingular operators the case is not exactly the same, as the canonical inclusion of c_0 in l_∞ shows, but in any case we have:

Theorem 2.3 *Let* $T : C(K, E) \longrightarrow F$ *be an operator whose representing measure* m *has semivariation continuous at* $\emptyset$. *Then* T *is strictly cosingular if and only if so is its extension* $\hat{T}$.

There are examples which prove that neither the conditions 1) and 2) of Corollary 2 nor the corresponding conditions for strictly cosingular operators characterize the strictly singular or the strictly cosingular operators between $C(K, E)$ and F. However, we can obtain a characterization of that operators if we assume some aditional conditions on the representing measure.

Theorem 2.4 *Let* K *be a compact dispersed space,* $T : C(K, E) \longrightarrow F$ *an operator and* m *its representing measure. The following assertions are equivalent:*

1. T *is strictly singular.*
2. $|m|$ *is continuous at* $\emptyset$ *and for each* $A \in \beta_0(K)$ *the operator* $m(A)$ *is strilly singular.*

Theorem 2.5 *Let* K *be a compact dispersed space, and* $T : C(K, E) \longrightarrow F$ *an operator whose representing measure* m *has semi-variation continuous at* $\emptyset$ *and such that for each Borel set* $A \subseteq K$, *the map* $m(A)$ *is strictly cosingular. Then* T *is strictly cosingular.*

In particular

Corollary 2.6 *Let* K *be a compact dispersed space and let* E, F *be any two different spaces of the set* $\{c_0\} \cup \{l_p : 1 \leq p < 0\}$, *with* $F \neq c_0$

1. *Any opperator* $T : C(K, E) \longrightarrow F$ *is strictly singular.*
2. *If* $E \neq l_1$, *any operator* $T : C(K, E) \longrightarrow F$ *is strictly cosingular.*

Remark:
Every separable Banach space F is isomorphic to a quotient of l_1. Hence there are always a surjection from l_1 onto F, and so we have to exclude $\mathrm{E} = l_1$ in part (b) of the corollary. On the other hand, if K is infinite and E of infinite dimension, $C(K, E)$ contains always a complemented copy of c_0 ([2]), which shows that $F = c_0$ has to be excluded also.

We study the relation between the ideals of strictly singular or strictly cosingular operators and other ideals of operators, like the ideal of weakly compact operators, and, for the concrete case of $C(K, E)$ spaces, we have

Proposition 2.7 *Let* K *be a compact Hausdorff space.*

1. *If* E *is reflexive, every strictly singular operator on* $C(K, E)$ *is weakly compact.*
2. *If* $E = c_0, l_1$ *or* $C(S)$ *(*S *a compact Hausdorff space), every weakly compact operator on* $C(K, E)$ *is strictly singular.*

3 APPLICATIONS.

In this applications, K will always be a compact dispersed space. In the first place, the following proposition extend a well known result of Pelczynski for $C(K)$ spaces ([7])

Proposition 3.1 *Let E be a Banach space such that every infinite dimensional subspace contains a copy of c_0 (i. e. E is hereditarily c_0). Then, every infinite dimensional complemented subspace of $C(K,E)$ contains a copy of c_0.*

The key of the proof is that the existence of a non strictly singular operator from $C(K,E)$ to F whose representing measure has semivariation continuous at $\emptyset$ implies the existence of a non strictly singular operator from E to F. We used this fact to set some information about the structure of subspaces of $C(K,E)$. For example, we obtain the following result, that extends in some sense a previous one of E. and P. Saab ([8]).

Proposition 3.2 *If $1 \leq p < \infty$, the following assertions are equivalent:*

1. *$C(K,E)$ contains a complemented copy of l_p.*
2. *E contains a complemented copy of l_p.*

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