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CARDANO TYPE ENTIRE FUNCTIONS

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Let Q a polynomial of degree q , normalized by Q(0)=1 and with roots $[w_n]_{n=1}^{q}$. Cardano's formulas allow us to represent Q in the form

$$Q(z) = \prod_{n=1}^{q} \left(1 - \frac{z}{w_n}\right) = \sum_{k=0}^{q} (-1)^k D_k z^k$$

where $D_0 = 1$ and $D_k = \sum_{1 \le i_1 < \ldots < i_k \le q} w_{i_1}^{-1} \ldots w_{i_k}^{-1}$ for $k \le q$.

The main subject in this paper is to extend this situation to entire functions , that is to study when

$$(1) \qquad f(z) = \prod_{n \ge 1} \left(1 - \frac{z}{w_n}\right) = \sum_{k \ge 0} (-1)^k D_k z^k$$

is verified , being $\Lambda = [w_n]_{n=1}^{+\infty}$ the zeros of the entire function f , normalized by f(0)=1 and

$$D_k = \lim_n \sum_{1 \le i_1 < \ldots < i_k \le n} w_{i_1}^{-1} \ldots w_{i_k}^{-1}$$

Several questions arise when we consider this problem : (i) the convergence in $\mathbb{C}$ of the infinite product $\Pi(1 - z w_n^{-1})$. (ii) the existence of D_n and convergence of the series $\Sigma (-1)^k D_k z^k$. (iii) conditions about f so that (1) can be satisfied . This study is connected with a particular type of multiplier sequences of the first kind of Laguerre-Pólya .

In what follows, $\Lambda = [w_n]_{n=1}^{+\infty}$ stands for an unbounded set of complex numbers no zeros , with no limit points and rank m .

DEFINITION 1 $\Pi(1 - z w_n^{-1})$ is said to be an **infinite product Cardano type** (denote $\Lambda \in CT$) , if it coincides with $\Sigma (-1)^k D_k z^k$

and defines an entire function.
This definition leads us firstly to study the existence of the D_k, and it is connected with the series $B_j = \Sigma w_n^{-j}$.

LEMMA

D_k exists for every k if only if B_k exists for every k. Moreover

$$D_{k+1} = \frac{(-1)^k}{k+1} \left[B_{k+1} + \sum_{l=1}^{k} (-1)^l D_l B_{k+1-l} \right]$$

It is an infinite version of the Newton-Girard formula.

THEOREM 1

$\Lambda \in CT$ if and only if every B_k exists for $1 \le k \le m$.

The existence of B_k, $1 \le k \le m$, is equivalent to that $\Pi (1 - z w_n^{-1})$ converges uniformly on compacts, and also, it turns out to be equivalent to verify that $\log [\Pi (1 - z w_n^{-1})] = (-1) \Sigma B_k z^k/k$ in $D(0,R)$, being log the principal branch of the logarithm.

Below we give some geometrical conditions on the set $\Lambda = \{w_n\}_{n=1}^{+\infty}$ in order to obtain infinite products Cardano type.

COROLLARY 1

(i) If $\Lambda \subset S$, where S is a closed sector with vertex in the origin and amplitude smaller than π, and B_1 exists, then $\Lambda \in CT$ and

$$\prod_{n \ge 1} \left(1 - \frac{z}{w_n} \right) \text{ converges absolutely.}$$

(ii) If $\Lambda \subset S^*$, where S^* is a closed sector with vertex in the origin and amplitude smaller or equal than π, and B_1, B_2 exist then $\Lambda \in CT$.

(iii) Let Λ be with $0 < |w_1| < |w_2| < \ldots$. If B_j, $1 \le j \le k$, exist and α, β are real numbers such that $0 < \alpha - \beta < 2\pi k/k+1$ and $\alpha \le \arg w_n^{-1} < \beta$, $n=1,2,\ldots$, then $\Lambda \in CT$.

Next we indicate a method to obtain CT sets begining from $\Lambda = \{w_n\}_{n=1}^{+\infty}$. Let p be such that $2^{p-1} \le k \le 2^p$ and denote by α_n, $n=1,2,\ldots,2^p-1$, the 2^pth roots of unity. We define $a_k = w_1 \alpha_{k-1}^{-1}$ if $1 \le k \le 2^p$ and $a_k = w_{n+1} \alpha_r^{-1}$ if $k > 2^p$, where $k = n 2^p + r$. Then $\{a_k\} \in CT$.

In what follows let f stand for an entire function with finite exponential order ρ (exp.ord. $f = \rho$) normalized by $f(0)=1$, and with rank m, i.e., $m = \text{rank } Z_f$ being Z_f the set of zeros of f.

DEFINITION 2 We say that f is **Cardano type** (denote $f \in CT$), if there is a rearrangement $\{w_n\}_{n=1}^{+\infty}$ of Z_f such that

$$f(z) = \prod_{n\geq 1} \left(1 - \frac{z}{w_n}\right) = \sum_{k\geq 0} (-1)^k D_k z^k$$

Denote $H(z) = \log [\Pi (1 - z w_n^{-1})]$ where log is the principal branch of the logarithm in $D(0,R)$.

THEOREM 2

$f \in CT$ if and only if there is a rearrangement of set Z_f such that $H^{(k}(0) = (-1)(k-1)!\, B_k$ for $k=1,2,\ldots$

If $\{\gamma_k\}_{k=0}^{+\infty}$ is a multiplier sequence ($\gamma_0=1$) such that $F(z) = \Sigma\, \gamma_k z^k/k!$ is an entire function of exponential order smaller than one, then $F \in CT$ and $D_k = (-1)^k \gamma_k/k!$ for $k\geq 0$. The zeros of F can be calculated by means of an iterative process.

We give some results related to derivatives of Cardano type functions, observing that they have, in some cases, analogous properties to the polynomials.

COROLLARY 2

(i) If f is an even or odd function and exp.ord. $f < 2$, then $f^{(k} \in CT$.

(ii) Let $f \in CT$ and $Z_f \subset \mathbb{R}$. Then:

(a) $Z_{f'} \subset \mathbb{R}$ and rank $f' \leq 1$. Moreover, between two simple zeros of f there is one and only one of f', and it turns out to be for every derivative when the exponential order of f is not a natural number.

(b) exp. ord. $f \leq 2$ and $f^{(k}$ takes real values on $\mathbb{R}$ for every k.

(c) If exp.ord. < 2, $f^{(k} \in CT$ for every k.

(d) If exp.ord.$f=2$, there exists $A \in \mathbb{R}$ such that $f^{(k}(z)\exp(-Az^2) \in CT$.

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