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A NOTE ON HALLEY'S METHOD

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In this paper , we study the influence that the convexity of a real function f has in Halley's method [4 , 5] , in order to get the solution of $f(x) = 0$. References [2] and [3] give global convergence theorems of this method .

For each convex function we introduce an index . This is the number of times that we need to compose the function with the logarithmic function in order to get a concave one . This concept , called degree of logarithmic convexity , provides a measure of the convexity of f at each point .

Let $f \in C^{(2}(V)$ a convex and positive function , V a neighbourhood of $x_0 \in \mathbf{R}$. Denote $T[f](x) = f(x) - f(x_0) + 1$ and define $F_1 = \log f$, $G_1 = T[F_1]$. By induction , if $n \in \mathbf{N}$: $F_{n+1} = \log G_n$ and $G_{n+1} = T[F_{n+1}]$. If x_0 is not a minimum of f , it is inmediate that F_n is a convex function if and only if $f(x_0) f''(x_0) [f'(x_0)]^{-2} \geq n$. It leads us to introduce the following

DEFINITION

The **degree of logarithmic convexity** of f at x_0 is defined by

$$(1) \qquad L_f(x_0) = \frac{f(x_0) f''(x_0)}{[f(x_0)]^2}$$

Notice that if x_0 is a minimum of f then F_n is convex for all n and so $L_f(x_0) = +\infty$. We can extend formally this definition for every function $f \in C^{(2}(V)$.

In terms of the degree of logarithmic convexity , Halley's method consists in applying the iterative process given by

$$(2) \quad x_n = F(x_{n-1}) \quad \text{with} \quad F(x) = x - \frac{f(x)}{f'(x)} \, \frac{2}{2 - L_f(x)}$$

It is known [1], that Halley's method can be derived by applying Newton's method to the function

$$h(x) = \frac{f(x)}{[f'(x)]^{1/2}}$$

In what follows ,we take f satisfying the following conditions

(3) $f \in C^{(3}([a,b])$, $f(a)<0<f(b)$, $f'(x)>0$ and $f''(x) \geq 0$ for $x \in [a,b]$

These conditions imply that there exists one and only one root $s \in (a,b)$ of the equation $f(x) = 0$. Suppose that the starting value x_0 satisfies $s \leq x_0 \leq b$. If we study the convergence of Newton's method for the function h , by means of the degree of logarithmic convexity , we obtain a new theorem of global convergence for Halley's method .

<u>THEOREM</u>

(i) If $L_{f'}(x) \leq 3/2$ in [a,b] then $\{x_n\}$, given by (2) , is a decreasing sequence that converges to s .

(ii) If $L_{f'}(x) \in (3/2, 2)$ and $L_f(x) < 1$ in [a,b] , for $x_0 \geq a + 2f(b)/f'(a)$, then the sequence $\{x_n\}$, given by (2), converges to s.

Proof :

(i) It is inmediate that h has a point of inflexion at the root s since

$$h''(x) = \frac{f'(x)}{4[f'(x)]^{1/2}} L_f(x) \quad [3 - 2L_{f'}(x)]$$

Then , if $L_{f'}(x) \leq 3/2$, it follows that h is a concave function in (a,s) and a convex function in (s,b) . On the other hand , as h'' is a positive function in (s,b) then $h'(x) = f'(x)^{1/2}(2 - L_f(x))/2$ is an increasing function in (s,b) . Besides , $h'(s) > 0$ and therefore h is an increasing function in (s,b) . By applying Newton's method to h , we obtain that $\{x_n\}$ is a decreasing sequence that

converges to s.
(ii) L_h is a negative function in $[a,b]$ and $|L_h(x)| < 1$ if and only if $f'(x)(2 - L_f(x)) - f(x)L_f'(x) > 0$ Then

$$L_f'(x) = \frac{f''(x)}{f'(x)} [1 + L_f(x)(L_{f'}(x) - 2)]$$

Therefore, we obtain that $|L_h(x)| < 1$ if and only if $2 - L_f(x) > L_f(x)[1 + L_f(x)(L_{f'}(x) - 2)]$. Then, taking into account that $L_f(x) \in (3/2,2)$ and $L_f(x) < 1$ in $[a,b]$ it follows that $|L_g(x)| < 1$ in $[a,b]$. Thus there exists $M \in (0,1)$ such that $|L_h(x)| < M$ in $[a,b]$.

On the other hand, if we denote $H(x) = x - \frac{h(x)}{h'(x)}$ and $x_n = H(x_{n-1})$, it is inmediate that $H(x_0) \in (a,s)$ and $|x_1 - s| < M |x_0 - s|$ since $x_1 - s = H(x_0) - H(s) = H'(\xi_0)(x_0 - s)$ for $\xi_0 \in (s,x_0)$. By induction, we obtain that $H(x_{2n}) \in (s,b)$ and $H(x_{2n+1}) \in (a,s)$ for $n \geq 0$. Besides $|x_n - s| < M^n |x_0 - s|$ and therefore $\lim_n x_n = s$. #

If the function f is decreasing, all the previous results turn out to be valid by changing slightly the reasoning used. The condition $f(x_0) > 0$ does not affect the results When $f(x_0) < 0$ and $L_{f'}(x) \leq 3/2$, $\{x_n\}$ turns to be an increasing sequence. If f is concave, then by considering the respective degree of exponential concavity, we obtain analogous results.

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