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ON STRONG SMALL TRANSFINITE DIMENSION AND D-DIMENSION

by

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In [6], D.W. Henderson introduced the **D-dimension** for metrizable spaces as a transfinite extension of the large inductive dimension; for non-metrizable spaces, the D-dimension has been studied in [2] and [7]. Also, in [1] P. Borst introduced the **strong small transfinite dimension** (sind) by extending the concept of ω_1-strongly countable dimensional spaces defined in [8]. Spaces that have strong small transfinite dimension are studied by Borst ([1]) and Hattori ([4]); the last author, following L. Polkowski ([7]), states that for strongly hereditarily normal spaces the existence of a strong small transfinite dimension and a D-dimension are equivalent. Also, Polkowski states in [7] that in the realm of the strongly hereditarily normal spaces, the spaces having a D-dimension coincide with the small spaces defined therein.

In this paper we establish some properties of the strong small transfinite dimension and the D-dimension for certain perfectly normal spaces; such spaces are strongly hereditarily normal and so, if X is perfectly normal, the existence of sind(X) is equivalent to the one D(X). In order to characterize these spaces, we introduce the wf-Baire spaces and the w-strongly ordinally dimensional spaces as a cardinal generalization of the Baire spaces and the strongly countable dimensional spaces respectively which depends of the weight of the space. Also, we give a comparison theorem for these two transfinite dimensions in the class of the perfectly normal and paracompact spaces.

Definition 1. *We say that a topological space X is a wf-Baire space if no non-empty open set G in X can be expressed as a union of a* *transfinite sequence of sets* $\{A_\alpha \mid 0 \leq \alpha < \mu\}$ *with* $\overset{\circ}{A}_\alpha = \emptyset$ *for every* $\alpha < \mu$, *and* $|\mu| \leq w(X)$.

If X is a w-Baire space with $w(X) \leq \aleph_0$, then X is a Baire space. However, there are Baire spaces which are not w-Baire:

Example 1. Let R be the set of real numbers with the topology $T_{CF} = \{U \subset R \mid R \setminus U$ is finite$\}$. One easily checks that (R, T_{CF}) is a Baire space; however it is not a w-Baire space since $|R| \leq w(R, T_{CF})$ and $R = \bigcup \{\{x\} \mid x \in R\}$ with $\overset{\circ}{\overline{\{x\}}} = \emptyset$.

The w-Baire spaces are hereditary for open subspaces. However, they are not hereditary for closed subspaces:

Example 2. The space R of the real numbers with the topology T induced by the sub-basis:

$$S = \{U \subset R \mid U \in T_u\} \cup \{R \setminus Q\}$$

is a Baire space and it is second countable; therefore, it is a w-Baire space. However, the set Q of the rational numbers is a closed set of (R, T) which is not a w-Baire subspace of (R, T).

Definition 2. *A wf-Baire space is a topological space such that all its closed subspaces are w-Baire spaces.*

Definition 3. *Let X be a normal space which weight is* $w(X)$; *we say that X is w-strongly ordinally dimensional if every closed set C of X is the union of a transfinite sequence* $\{X_1, X_2, \ldots, X_\alpha, \ldots\}$ $(\alpha < \mu)$ *of closed subspaces of X such that* $Ind(X_\alpha) < \infty$ *for every* $\alpha < \mu$ *and* $|\mu| \leq w(C)$.

Obviously, if X is a second countable normal space, X is w-strongly ordinally dimensional if and only if it is strongly countable dimensional.

There exists normal spaces which are w-strongly ordinally dimensional while they are not strongly countable dimensional:

Example 3. The space X defined in [3] (example 3):

$$X = \bigcup_{i=1}^{\infty} \bigcup_{m=0}^{i} [B(S_i^m) x (R_i^m \cup P)] \subset \widetilde{B}(\omega_1) x Z$$

is a perfectly normal space which is not second countable nor strongly countable dimensional. Every closed set C of the space X can be represented as:

$$X = \bigcup \{[B(\alpha) x Z] \cap X \mid \alpha < \lambda\}$$

where λ is a countable ordinal number or $\lambda=\omega_1$ if $w(C)\leqslant\aleph_0$ or $w(C)\leqslant\aleph_1$ and $[B(\alpha)xZ]\cap X$ is the countable union of 0-dimensional sets.

Now, we have:

Theorem 1. *For a perfectly normal, paracompact, wf-Baire space X we have $D(X)<\Delta$ or X has a strong small transfinite dimension if and only if X is a w-strongly ordinally dimensional space.*

Corollary. *For every perfectly normal, paracompact, wf-Baire space X the following conditions are equivalent:*

a) $D(X)<\Delta$.

b) X has a strong small transfinite dimension

c) X is a w-strongly ordinally dimensional space.

d) X is a strongly countable dimensional space.

Theorem 2. *If X is a perfectly normal, weakly paracompact space, then $D(X)\leqslant sind(X)$.*

Corollary. *If X is a metrizable space, $D(X)\leqslant sind(X)$.*

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