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Lattice isomorphisms of alternative algebras * +

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Abstract: The lattice of subalgebras of an alternative algebra can determine the algebraic structure of the algebra. Here, it is showed that, alternative algebras with lattice of subalgebras isomorphic to a non division semisimple alternative algebra, are closely related with them.

Introductión

Let A be an alternative algebra over the field F. It is known the set of subalgebras of A has a lattice structure. We denote this lattice by $\mathfrak{L}(A)$. Let B be another algebra over the field F. By an $\mathfrak{L}$-isomorphism, or lattice isomorphism of the algebra A onto an algebra B, we mean a one to one map Ψ: $\mathfrak{L}(A) \longrightarrow \mathfrak{L}(B)$ such that $\Psi(A_1 \vee A_2) = \Psi(A_1) \vee \Psi(A_2)$ and $\Psi(A_1 \cap A_2) = \Psi(A_1) \cap \Psi(A_2)$, for all A_1 and A_2 subalgebras of A, where we denote by $A_1 \vee A_2$ the least subálgebra of A containing A_1 and A_2.

We are interested in the study of the lattice isomorphisms of alternative algebras. We want to inquire into the algebraic relationships between a semisimple alternative algebra and an alternative algebra $\mathfrak{L}$-isomorphic to it.

Here, we solve the problem when A is a simple non division finite dimensional algebra, that is, if A is a matrix algebra, $M_n(D)$, with $n \geq 2$ and D a division associative algebra or if A is a split Cayley-Dickson algebra. Then, if $n \geq 3$, it is shown B is isomorphic or semiisomorphic to A (If $n=2$, B-

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$M_2(\Delta)$ with Δ division associative algebra).When A is a division central non associative algebra, that is, a division Cayley-Dickson algebra over F, it is shown B must be a division Cayley-Dickson algebra or a purely inseparable p^3-dimensional extension field of F, with p characteristic of F. In case A is a finite dimensional semisimple algebra, we extend the Barne's results for associative algebras [5], and we show B is also semisimple and the images under Ψ of simple direct summands of A, with dimension bigger than one, are simple direct summands of B.

We always consider simple and semisimple algebras with finite length. It is clear that $\dim_F(A) \geq l(A)$.

§1 : $\mathcal{L}$-isomorphic alternative algebras to a central Cayley-Dickson algebra and an alternative nondivision simple algebra.

Theorem 1.1: *Let A be an alternative algebra over F, field with char F $\neq 2$, $\mathcal{L}$-isomorphic to a central division Cayley-Dickson algebra. Then A is a division central Cayley-Dickson algebra or if char $F = p > 0$ a purely inseparable p^3-dimensional extension field of F.*

Corollary 1.2: *Let F be a perfect field and Ψ: $\mathcal{L}(C) \rightarrow \mathcal{L}(A)$ $\mathcal{L}$-isomorphism of alternative algebras over F. If C is a division central Cayley-Dickson algebra, then A is a division central Cayley-Dickson algebra.*

Theorem 1.3: *Let $S = M_n(\Delta)$ where $n \geq 2$ and Δ is a finite dimensional division associative algebra. Let A be an algebra $\mathcal{L}$-isomorphic to S by the $\mathcal{L}$-isomorphism " Ψ ". Then $A \cong M_n(D)$ where D is division associative algebra $\mathcal{L}$-isomorphic to Δ such that $d(D) = d(\Delta)$.*

Theorem 1.4: *Let $S = M_n(\Delta)$ with Δ finite dimensional division algebra and $n \geq 3$. Let A be an alternative algebra $\mathcal{L}$-isomorphic to S. Then S is semiisomorphic to A.*

Theorem 1.5: *Let C be a split Cayley-Dickson algebra over K, extension field of F, and let A be an alternative algebra $\mathcal{L}$-isomorphic to C. Then A is isomorphic to C.*

<u>§ 2 : Alternative algebras $\mathcal{L}$-isomorphic to a semisimple algebra.</u>

In the following the proofs are like in the associative case using the study about the simple alternative case already made.

<u>*Theorem 2.1:*</u> *Let A be a finite-dimensional semisimple algebra over the field F, and let* $\Psi : \mathcal{L}(A) \longrightarrow \mathcal{L}(B)$ *be* $\mathcal{L}$*-isomorphism. Let* $S_1, \ldots, S_r$ *be the simple direct summands of A. Suppose A is not a division algebra and, in the case F is not the field of two elements, that not all* S_i *are one-dimensional. Then B is semisimple. For each* S_i *with* $\dim_F(S_i) > 1$, $\Psi(S_i)$ *is a simple direct summand of B. If* $S_i \cong S_j$*, then* $\Psi(S_i) \cong \Psi(S_j)$.

<u>*Corollary 2.2*</u> *Let A be a finite-dimensional semisimple algebra over an algebraically closed field F. Suppose* $\dim_F(A) > 1$*. Let* $\Psi: \mathcal{L}(A) \longrightarrow \mathcal{L}(B)$ *be* $\mathcal{L}$*-isomorphism. Then* $A \cong B$.

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