EXTRACTA MATHEMATICAE 3, n.3, 119-121 (1988)

Alternative Algebras which are L-isomorphic to a central division Cayley-Dickson algebra of characteristic 2.

J. A. Laliena

C.U.R. (University of Zaragoza)
c/ Obispo Bustamante nº 3
26001. Logroño. Spain.

A.M.S. Classification 17D05

Abstract: Alternative algebras which are L-isomorphic to a central division quaternion algebra of characteristic 2, and to a central division Cayley-Dickson algebra of characteristic 2 are studied. In the second case, we show they are purely inseparable extension fields with dimension 8 or central division Cayley-Dickson algebras.

Introduction

Alternative algebras which lattice of subalgebras is L-isomorphic to the lattice of subalgebras of a semisimple non division alternative algebra, and to a central division Cayley-Dickson algebra of characteristic not two were already studied in [3]. Here we complete [3], showing what happens with central division Cayley-Dickson algebras with characteristic two.

We recall an alternative algebra is a not necessarily associative algebra that verifies the following identities:

$$x^2 y - x(xy) \qquad y x^2 - (yx)x \qquad (1)$$

From (1) we remark that associative algebras are alternative. But there are alternative algebras that are not associative. For example simple alternative algebras are associative or Cayley-Dickson algebras [1], [2], and the last ones are not associative.

In the following by an alternative algebra we mean an alternative algebra over a field F with characteristic 2 and finite dimensional.

The map Φ from the lattice of subalgebras of A onto the lattice of subalgebras of B, denoted by L(A) and L(B) respectively, is said L-isomorphism if it is a one-to-one map and it verifies:

$\Phi(C \vee D) = \Phi(C) \vee \Phi(D)$ and $\Phi(C \cap D) = \Phi(C) \cap \Phi(D)$

for all C, D $\leq$ A and where CvD=the least subalgebra of A containing C and D.

Now we show the table of the basis of a central Cayley-Dickson algebra C over a field F with characteristic 2, where the element 1_C is omitted

	v_1	v_2	v_2v_1	v	vv_1	vv_2	$v(v_2v_1)$
v_1	$v_1+\mu$	$v_2(1+v_1)$	μv_2	vv_1	μv	$v(v_2v_1)$	μvv_2
v_2	v_2v_1	β	βv_1	vv_2	$v(v_2v_1)$	βv	βvv_1
v_2v_1	$v_2(v_1+\mu)$	$\beta(1+v_1)$	$\beta\mu$	$v(v_2v_1)$	$v(v_2v_1+\mu v_2)$	$v(v_2+\mu)$	$\beta(vv_1+\mu v)$
v	vv_1	vv_2	$v(v_2v_1)$	γ	γv_1	γv_2	γv_2v_1
vv_1	$v(v_1+\mu)$	$v(v_2v_1)$	$v(v_2v_1+\mu v_2)$	γv_1	$\gamma(\mu+v_1)$	$\gamma(v_2+v_2v_1)$	$\gamma\mu v_2$
vv_2	$v(v_2+v_2v_1)$	βv	$\beta(v+vv_1)$	γv_2	$\gamma(v_2+v_2v_1)$	$\gamma\beta$	$\gamma(\beta+\beta v_1)$
$v(v_2v_1)$	μvv_2	βvv_1	$\beta\mu v$	γv_2v_1	$\gamma\mu v_2$	$\gamma\beta v_1$	$\gamma\beta\mu$

From [4] it is known that maximal subalgebras of C are quaternion algebras. Thus, for study alternative algebras L-isomorphic to a central division Cayley-Dickson algebra with characteristic 2, we need to know first alternative algebras, A, L-isomorphic to Q, central quaternion division algebra with characteristic 2 (then Q has a basis $\{1, v_1, v_2, v_2v_1\}$). Because of [3] A will be a nilpotent algebra or a division algebra.

<u>Lemma1:</u> *A nilpotent alternative algebra, A, is L-isomorphic to Q if and only if A is three dimensional over F and has a basis over F, $\{a, b, c\}$, such that his multiplication table is given by:* $a^2 = ab = c$ $\quad ba=0$ $\quad b^2=\gamma c$ *with* $\gamma \in F-\{0\}$*, and moreover F must verify that the equation* $X^2+X+\gamma=0$ *has not solution in F.*

-Proof- (see [3])

<u>Lemma 2:</u> *Let A be a division alternative algebra L-isomorphic to Q. Then A is a purely inseparable eight dimensional extension field of F or a division central quaternion algebra.*

-Proof-

Subalgebras of Q with length two will be purely inseparable extension fields of F or separable extension fields of F with dimension 2, such that two of them, which are different have intersection F and they span all Q.

In the same way than [3] we can show now A is a purely inseparable extension field with dimension eight or A is m^2-dimensional division associative algebra with length three and where m is the dimension of the subfields of A with length two (which must be purely inseparable extension fields of F or separable extension fields of F).

Since characteristic of F is 2, we will have m=2 and therefore dimension of A is 4. But then A is a simple quadratic algebra. From [5] A will be a central division quaternion algebra.

Theorem: *If A is an alternative algebra L-isomorphic to a central division Cayley-Dickson algebra with characteristic 2, then A is a purely inseparable extension field of F with dimension eight or a division central Cayley-Dickson algebra.*

-Proof-

Apply [1] and this fact: In C the subalgebras Q_1, Q_2 with F-basis $\{1, v_1, v_2, v_2v_1\}$, $\{1, v, v_1+vv_2, vv_1+\gamma v_2\}$ have the following property: $Q_1 \cap Q_2 = F$.

Corollary: *If F is a perfect field and A is an alternative algebra L-isomorphic to a central division Cayley-Dickson algebra, then A is also a central division Cayley-Dickson algebra.*

REFERENCES

[1] E. Kleinfeld: *"Alternative division rings of characteristic 2"* Proc. Natl. Acad. Sci. Vol 37, No. 12, pp 818-820 . December 1951.

[2] E. Kleinfeld: *"Simple alternative rings"* Ann. of Math. (2) 58 (1953) p.p. 544-547.

[3] J. Laliena: *"Lattice isomorphisms of alternative algebras"* J. of Algebra (to appear).

[4] M.L. Racine: *"On Maximal subalgebras"* J. of Algebra 30 (1974) p.p. 155-180.

[5] K.A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, A. I. Shirshov: *"Rings that are nearly associative"* Academic Press. New York. (1982).