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A NOTE ON RELATIVE F.B.N. RINGS

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It is a well known fact that in commutative noetherian rings there exists a bijection between the prime ideals of the ring and the isomorphism classes of indecomposable injective modules (e.g. see [7]). In the noncommutative case, in general, this bijection no longer holds. Thus arises the study of the so called fully bounded noetherian rings.

The torsion-theoretic generalization of noetherian rings, i.e. the $\mathfrak{F}$-noetherian rings, where $\mathfrak{F}$ is a Gabriel filter, has widely been studied (for example see [2], [7] and their references).

In [1] a torsion-theoretic generalization of the fully bounded rings is given, assuming the $\mathfrak{F}$-noetherian condition, that is, the ascending chain condition for the $\mathfrak{F}$-saturated left ideals. This new class of rings includes examples so interesting as $\mathfrak{F}$-artinian rings, (see [2] for more details), commutative $\mathfrak{F}$-noetherian rings and some extensions of commutative rings by central algebras and stable Gabriel filters, studied in [6]. Concretely, let R be a commutative ring and Λ an unital central R-algebra. Let $\mathfrak{F}$ be a stable Gabriel filter over R and let $\mathfrak{F}^{\bullet}$ be the filter $\{I \subseteq \Lambda,\ (\Lambda/I)_p = 0,\ \forall\ p \in \mathrm{Spec}_{\mathfrak{F}}(R)\}$. Assume that for every $p \in \mathrm{Spec}_{\mathfrak{F}}(R)$, R_p is noetherian and Λ_p is a finite R_p-algebra. Then there exists a bijection between the isomorphism classes of $\mathfrak{F}^{\bullet}$-torsionfree indecomposable injective Λ-modules and the $\mathfrak{F}^{\bullet}$-saturated prime ideals of Λ. Similar results are obtained also working in separable and central algebras.

In this note we give a characterization of these rings in terms of its relative Krull dimension, introduced by Jategaonkar in [5]. As it is well-known, every noetherian module has Krull dimension. It seems to be reasonable asking for the relationship between the $\mathfrak{F}$-noetherian condition and the relative Krull dimension, and, if it is possible, using this machinery of relative Krull dimension for the study of $\mathfrak{F}$-noetherian rings.

Throughout this paper, we follow the notation and terminology of [6]. By following Jategaonkar, for a module M, the relative Krull dimension of M, denoted $K_{\mathfrak{F}}$-dim(M), is defined inductively as follows: If M is $\mathfrak{F}$-torsion then $K_{\mathfrak{F}}$-dim(M) = -1. If M is $\mathfrak{F}$-artinian then $K_{\mathfrak{F}}$-dim(M) = 0. If

α is an ordinal and $K_{\mathfrak{F}}$-dim(M) $\nleq \alpha$, then $K_{\mathfrak{F}}$-dim(M) $= \alpha$ provided there is no infinite descending chain $M_{\mathfrak{F}} = X_0 \supseteq X_1 \supseteq \ldots$ of subobjects X_i of $M_{\mathfrak{F}}$, the localized of M in the quotient category $(R,\mathfrak{F})$-Mod, such that for i = 1, 2, ... $K_{\mathfrak{F}}$-dim$(X_{i-1}/X_i) \nleq \alpha$.

We define $K_{\mathfrak{F}}$-dim(R) = $K_{\mathfrak{F}}$-dim$({}_R R)$. A module M, which is not $\mathfrak{F}$-torsion, is said to be *α-critical with respect to $\mathfrak{F}$* if $K_{\mathfrak{F}}$-dim(M) $= \alpha$ and $K_{\mathfrak{F}}$-dim(M/N) $< \alpha$ for each $\mathfrak{F}$-saturated submodule N which properly contains t(M).

The proofs of Theorem 2.1 and Proposition 1.3 of [4] remain valid in the above setting. Thus any $\mathfrak{F}$-noetherian module has relative Krull dimension with respect to $\mathfrak{F}$, and every module with relative Krull dimension contains a relative α-critical module. If the module, in addition to having relative Krull dimension is not $\mathfrak{F}$-torsion, then we can assert that such relative α-critical submodule is also $\mathfrak{F}$-saturated. These assertions are proved in [3].

Also in [3], it is proved that the $\mathfrak{F}$-torsionfree indecomposable injective R-modules, where R is a ring with relative Krull dimension, are the injective hulls of the $\mathfrak{F}$-torsionfree relative α-critical modules. Thus, if E = E(C) is a $\mathfrak{F}$-torsionfree indecomposable injective, with C $\mathfrak{F}$-torsionfree and relative α-critical, we define the <u>relative critical dimension</u> of E, denoted by $Cr_{\mathfrak{F}}$-dim(E), as the relative Krull dimension of C. It is proved that this definition is independent of the chosen submodule C, because if C is a $\mathfrak{F}$-torsionfree and relative α-critical R-module, then $K_{\mathfrak{F}}$-dim(C) is minimal among relative Krull dimensions of nonzero submodules of E(C).

The next propositions show us the relationship between R and an arbitrary relative critical R-module in terms of its relative Krull dimension. They are the key in our purpose:

Proposition 1. *Let R be an $\mathfrak{F}$-torsionfree semiprime ring with $K_{\mathfrak{F}}$-dim(R) $= \alpha$ and assume that D is a relative α-critical R-module. Then D contains an isomorphic copy of a relative α-critical left ideal of R.*

Proposition 2. *Let R be an $\mathfrak{F}$-torsionfree prime ring with relative Krull dimension. Let C be an $\mathfrak{F}$-saturated and relative critical left ideal in R. Then $K_{\mathfrak{F}}$-dim(R) = $K_{\mathfrak{F}}$-dim(C).*

We use this machinery in the $\mathfrak{F}$-noetherian case and for $\mathfrak{F}$-torsionfree modules. In this case, we define the associated ideal of a $\mathfrak{F}$-torsionfree module as in the absolute case and everything works in a similar way. If the module is $\mathfrak{F}$-torsionfree and uniform, the associated ideal is an

$\mathfrak{T}$-saturated prime ideal (c.f. [2]).

If E is an $\mathfrak{T}$-torsionfree indecomposable injective R-module then it will be of the form E = E(C) with C $\mathfrak{T}$-torsionfree and relative critical. If R is $\mathfrak{T}$-noetherian, Ass(E) is an $\mathfrak{T}$-saturated prime ideal. We can choose a relative critical cyclic submodule C' such that ann(C') = P = ass(C). As C' is an R/P-module, we would have $K_{\mathfrak{T}}$-dim(C') ≤ $K_{\mathfrak{T}}$-dim(R/P).

Proposition 3. *Let R be an $\mathfrak{T}$-noetherian ring and $\mathfrak{P}$ an $\mathfrak{T}$-saturated prime ideal. Then there exists an $\mathfrak{T}$-torsionfree indecomposable injective R-module E, unique up to isomorphism, such that Ass(E) = $\mathfrak{P}$ and $Cr_{\mathfrak{T}}$-dim(E) = $K_{\mathfrak{T}}$-dim(R/$\mathfrak{P}$).*

Now, we can enounce our theorem of characterization:

Theorem 4. *Let R be an $\mathfrak{T}$-noetherian ring. Then the following statements are equivalent:*

1) The mapping E $\longrightarrow$ Ass(E) gives a bijection between isomorphism classes of $\mathfrak{T}$-torsionfree indecomposable injective R-modules and $\mathfrak{T}$-saturated prime ideals of R.

2) For each $\mathfrak{T}$-torsionfree indecomposable injective R-module E, we have $Cr_{\mathfrak{T}}$-dim(E) = $K_{\mathfrak{T}}$-dim(Ass(E)).

3) For each $\mathfrak{T}$-saturated prime ideal of R, the ring R/$\mathfrak{P}$ has the following property: each essential left ideal I/$\mathfrak{P}$ in R/$\mathfrak{P}$, where I is $\mathfrak{T}$-saturated, contains a nonzero two-sided ideal.

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