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On Grothendieck Space Ideals

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To Christine ("Wonderful Tonight") Glasson

An Operator Ideal is a class A of operators acting between Banach spaces and such that: 1.The finite-dimensional operators belong to A, 2.The sum of operators of A is in A, and 3.The composition of continuous operators with operators of A is in A (ideal property).

A locally convex space (lcs) E is said to be generated by the ideal A, or an A-space, if it can be represented as a reduced projective limit of Banach spaces with linking maps in A. The class of all A-spaces is noted $\mathcal{A}$=Groth(A), and called a Grothendieck space ideal. Grothendieck invented in |5| the two first classes of this kind: the Schwartz spaces=Groth(K), K being the ideal of compact operators, and the nuclear spaces=Groth(N), N the ideal of nuclear operators. There are many more.

It is not always easy to recognize a class M of lcs as a Grothendieck space ideal, that is, to find out which operator ideal A satisfies M=Groth(A). For instances, lcs carrying the weak topology = Groth(Finite-dimensional operators). This paper is centered on this problem. Mainly from a negative point of view.

Most classes appearing in the literature are Grothendieck space ideals. Only three exceptions seem to be known: Montel spaces |7|, Ω the class of nuclear spaces of maximal diametral dimension (equal to the minimal stability class generated by R) |4|, and the scales of $\Lambda_N(\alpha)$-nuclear spaces |6|. The proofs for these three classes seem to be completely unrelated.

The paper has three sections. In the first part hereditary and internal properties of Grothendieck space ideals are studied. They can be summed up as follows:

Theorem. Let $\mathcal{A}$ = Groth(A) be a Grothendieck space ideal. Then:

1. **$\mathcal{A}$ contains the product spaces K^I, I any set.**
2. **Dense subspaces and completions of A-spaces are A-spaces.**
3. **Arbitrary products of A-spaces are A-spaces.**
4. **Locally complemented subspaces of A-spaces are A-spaces.**
5. **$\mathcal{A}$ has enough metrizables.**

The notions of "local complementation" and "enough metrizables" are introduced in the paper:

Definition 1. A subspace E is said to be locally complemented in an lcs F when a fundamental system of neighborhoods of 0, U(F), can be found in F, such that for all $U \in U(F)$ the associated Banach space $\hat{E}_{U \cap E}$ is complemented in the associated Banach space $\hat{F}_U$.

Definition 2. We say that a class of lcs M possesses enough metrizables when for each $E \in M$ there exists a collection F_i, $i \in I$, of metrizable spaces in M such that E is locally complemented in $\prod_{i \in I} F_i$.

In the second part the above theorem is exploited to prove that some well-known classes of lcs appearing in the literature cannot be generated by an operator ideal (in fact for any class satisfying 1 and 3 in the definition of operator ideal, what is called an operator pre-ideal). Some of these classes are:

#1. The barrelled, $\aleph_0$, l_∞, or c_0-barrelled Schwartz, Schwartz-Hilbert or nuclear spaces.

#2. The Schwartz, Schwartz-Hilbert or nuclear spaces with the Bounded Approximation Property.

#3. The Schwartz, Schwartz-Hilbert or nuclear Δ-stable spaces.

Recall that an lcs E is said to be Δ-stable (|3|) if the diametral dimension of ExE equals the diametral dimension of E. Δ-stable spaces are of relative importance for the study of some topological properties of lcs, especially Schwartz spaces. A recent example is |1|.

For Montel spaces a new proof can be deduced from theorem 1, while for the class Ω it was already done in |2|.

The third section is devoted to the study of the class of $\Lambda_N(\alpha)$ nuclear spaces (in general a class of the form $\bigcap_{k \in N} \mathrm{Groth}(A_k)$. It was proved in |6| that the class of $\Lambda_N(\alpha)$-nuclear spaces is not a

Grothendieck space pre-ideal. This shows that that the five conditions of theorem 1 are not sufficient.

At the end of the paper a connection between diametral dimension, $\Lambda_N(\alpha)$-nuclearity and Δ-stability is established to show, among other things, that there exists a Fréchet space E such that for some sequence $a \in \Delta(E)$ and $a^2 \notin \Delta(E)$. This seems to be the first counterexample to |8|.

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