EXTRACTA MATHEMATICAE 4, n.1, 39-41 (1989)

IDEAL SEMIGROUPS ASSOCIATED TO AN OPERATOR IDEAL

M. Gonzalez, University of Santander, Spain.

AMS 1980 Classification: 47D30, 47A53, 46B20

In [3, 4, 5] semigroups associated with some operator ideals defined in terms of sequences were introduced. In this paper we associate to every operator ideal U two semigroups SU_+ and SU_- stable under perturbation by operators in U. If $T \in L(X,Y)$ is an operator and U injective (surjective), then the kernel N(T) (cokernel $Y/\overline{R(T)}$) of T belongs to the space ideal Sp(U).

For operator ideals verifying a condition related with the three space property [6, 7], we obtain characterizations for semi-Fredholm operators and some incomparability classes of Banach spaces [1]. We show that this condition is satisfied by the operator ideals considered in [3, 4, 5], and by the generalized strictly (co)-singular operators studied in [2, 9].

Recall that an operator $T \in L(X,Y)$ is <u>upper semi-Fredholm</u> ($T \in SF_+$) if it has closed range R(T) and finite dimensional kernel N(T). T is <u>lower semi-Fredholm</u> ($T \in SF_-$) if it has closed range and finite dimensional cokernel $Y/\overline{R(T)}$.

Following [8], an <u>injection</u> (<u>surjection</u>) will be an injective (surjective) operator with closed range.

Every operator ideal U has associated an <u>space ideal</u> [8]

$$Sp(U) := \{ X \ / \text{ the identity } I_X \text{ belongs to } U \}.$$

<u>Definition</u> Let U be an operator ideal and X,Y Banach spaces.

$SU_+(X,Y) := \{ T \in L(X,Y) \ / \ TA \in U \text{ implies } A \in U \}$

$SU_-(X,Y) := \{ T \in L(X,Y) \ / \ BT \in U \text{ implies } B \in U \}$

<u>Proposition</u> Let U be an operator ideal.

(1) SU_+ and SU_- are semigroups stable under perturbation by elements of U.

(2) If U is injective, then $SF_+ \subseteq SU_+$ and for every $T \in SU_+$ we have $N(T) \in Sp(U)$.

(3) If U is surjective, then $SF_- \subseteq SU_-$ and for every $T \in SU_-$ we have $Y/\overline{R(T)} \in Sp(U)$.

(4) If U = Co, then $SCo_+ = SF_+$ and $SCO_- = SF_-$.

Definition Let U be an operator ideal.

U is said to have the left three space property (L3SP) when given $A \in L(X,Y)$ and a surjection q such that $N(q) \in Sp(U)$ and $qA \in U$, then we have $A \in U$.

U is said to have the right three space property (R3SP) when given $B \in L(X,Y)$ and an injection i such that $X/R(i) \in Sp(U)$ and $Bi \in U$, then we have $B \in U$.

Note that the L3SP (or R3SP) implies the three space property: If I_M , $I_{X/M} \in U$ for some subspace M, then $I_X \in U$ [6].

Theorem Let U be an operator ideal containing the nuclear operators and $T \in L(X,Y)$.

(1) Suppose U is injective and verifies the L3SP. Then $T \in SF_+$ if and only if $T \in SU_+$ and for every subspace M of X in Sp(U) the restriction $Ti_M \in SF_+$.

(2) Suppose U is surjective and verifies the R3SP. Then $T \in SF_-$ if and only if $T \in SU_-$ and for every subspace N of Y such that $Y/N \in Sp(U)$ we have $q_N T \in SF_-$.

Theorem With the same hypothesis as in the above theorem,

(1) X belongs to $Sp(U)^i$ if and only if $SF_+(X,Y) = SU_+(X,Y)$ for every Banach space Y.

(2) X belongs to $Sp(U)^c$ if and only if $SF_-(Z,X) = SU_-(Z,X)$ for every Banach space Z.

Proposition The operator ideals of all compact, weakly compact, Rosenthal or l_1-singular, completely continuous and weakly completely continuous operators verify the L3SP, and the corresponding dual ideals verify the R3SP. Recall that $U^{dual} := \{ K / K' \in U \}$.

Given a subspace M of X, i_M will denote the natural injection of M into X and q_M the quotient map onto X/M.

Definition Let A be an space ideal and X, Y Banach spaces.

$A\text{-}SS(X,Y) := \{ T \in L(X,Y) / Ti_M$ injection implies $M \in A \}$

$A\text{-}SC(X,Y) := \{ T \in L(X,Y) / q_N T$ surjection implies $Y/N \in A \}$.

Recall that two Banach spaces X and Y are <u>totally incomparable</u> (<u>coincomparable</u>) [1] if Banach spaces isomorphic to a subspace (quotient) of X and to a subspace (quotient) of Y have finite dimension.

Given a space ideal A we shall denote A^i (A^c) the class of Banach spaces which are totally incomparable (coincomparable) with every space in A. A^i and A^c are space ideals, $A^i = A^{iii}$ and $A^c = A^{ccc}$ [1].

<u>Proposition</u> Let A be an space ideal.

(1) If $A = A^{ii}$, then A-SS is an operator ideal verifiyng the L3SP.

(2) If $A = A^{cc}$, then A-SC is an operator ideal verifiyng the R3SP.

We shall denote $S\mathbb{A}_+$ the semigroup SU_+ associated to $U = \mathbb{A}$-SS and $S\mathbb{A}_-$ the semigroup SU_- associated to U = A-SC.

<u>Theorem</u> Let $\mathbb{A}$ be an space ideal and $T \in L(X,Y)$.

(a) Suppose $\mathbb{A} = \mathbb{A}^{ii}$. $T \in S\mathbb{A}_+ \iff N(T+K) \in \mathbb{A}$ for every $K \in Co(X,Y)$ $\iff Ti_M \in SF_+$ for every subspace M of X in $\mathbb{A}^i$.

(b) Suppose $\mathbb{A} = \mathbb{A}^{cc}$. $T \in S\mathbb{A}_- \iff Y/\overline{R(T+K)} \in \mathbb{A}$ for every $K \in Co(X,Y) \iff q_M T \in SF_-$ for every subspace M of Y with $Y/M \in \mathbb{A}^i$.

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