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FINITELY ADDITIVE INTEGRATION II

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In this note we present some still unpublished result concerning finitely additive integration. We consider a nonnegative linear functional I (which in general is not continuous in eny sense), defined on a vector lattice B of real-valued functions on a set $X \neq \phi$, under the pointwise defined operations and relations $+$, $t.$, $=$, $\leq$, $\vee$, $\wedge$.

In [1] the initial elemental integral I is extended to a class B_o of *I-summable functions* by the previous introduction of the appropriate oscillation integrals $\bar{I}$ and $\underline{I}$.

For results and terminology we refere the reader to [1], [2] and [3].

Definition 1. (*$\bar{I}$-convergence*). Let f, $(f_n)_n \subset \bar{R}^X$, $f_n \to f(\bar{I})$ means to each $h \in B$, $h \geq 0$, $\varepsilon > 0$, there exists $n_o(\varepsilon, h) \in N$ such that $\bar{I}(|f_n - f| \wedge h) < \varepsilon$ if $n \geq n_o$.

Definition 2. The set $R(B,I)$ of *I-integrable functions* is defined as the set of all $f \in \bar{R}^X$, to which there exists a sequence $(h_n)_n \subset B$ which is an $\bar{I}$ Cauchy-sequence and with $h_n \to f(\bar{I})$.

If $f \in R(B,I)$, $\tilde{I}(f) := \lim I(h_n)$ as $n \to +\infty$.

$R(B,I)$ is an R-lattice and $\tilde{I}$ is linear on it.

For any $f \in R(B,I)$, $\| f \| := \tilde{I}(|f|)$ and B is $\|\cdot\|$-dense in $R(B,I)$.

Theorem 1. $R(B,I)$ is "closed" with respect to $\bar{I}$-convergence, i.e. let $f \in \bar{R}^X$, $(f_n)_n \subset R(B,I)$ I-Cauchy-sequence such that $f_n \to f\ (\bar{I})$, then $f \in R(B,I)$ and $\tilde{I}(/f_n - f/) \to 0$ as $n \text{ -- } +\infty$.

Corollary. (Monotone convergence theorem). Let $(f_n)_n \subset R(B,I)$, $f_n \leq f_{n+1}$, $n = 1,2,\ldots$, with $f_n \to f\ (\bar{I})$ and $\sup I(f_n) < +\infty$. Then, $f \in R(B,I)$ and $\tilde{I}(/f_n - f/) \to 0$ as $n \text{ -- } +\infty$.

Theorem 2 (Lebesgue's bounded convergence theorem). Let $(f_n)_n \subset R(B,I)$ such that $f_n \to f(\bar{I})$ and $/f_n/ \leq g \in R(B,I)$, $n = 1,2,\ldots,$. Then, $f \in R(B,I)$ and $\tilde{I}(/f_n - f/) \to 0$, as $n \text{ -- } +\infty$.

In the following some consequences are derived.

1.- $B_o := \{ f \in \bar{R}^X;\ \bar{I}(f) = \underline{I}(f) \in R \} \subsetneq R(B,I)$ and $I(f) := \bar{I}(f) = \tilde{I}(f)$, for all $f \in B_o$.

If $f \in R(B,I)$, then $f \in B_o$ iff $/f/ \leq g \in B_o$.

In particular, theorem 1 and theorem 2 can be formuled in B_o (see [1]).

2.- Let Ω be a semiring of subsets of X and μ a nonnegative finitely additive measure on Ω. B_μ denotes the set of all the step-functions and $I_\mu := \int \cdot\, d\mu$ as usual. Now, starting from I_μ and B_μ by using the above general methods, we obtain the class $R(B_\mu, I_\mu)$ of I_μ-integrable functions.

$R^1(\mu, \mathbb{R})$ denotes the set of all the abstract-Riemann-integrable functions of [5] .

In [2] it is shows that $R^1(\mu, \mathbb{R}) \subsetneq B_o + \{ f \in \bar{R}^X;\ \int f\, d\mu = 0 \}$.

Now, it has been shown that in $R^1(\mu, \mathbb{R})$ convergence μ-locally of [5] and $\bar{I}$-convergence both coincide, therefore we have the following result:

$R^1(\mu, \mathbb{R}) \subsetneq R(B_\mu, I_\mu)$ and $\int f\, d\mu = \tilde{I}(f)$ for all $f \in R^1(\mu, \mathbb{R})$. Thus, the lebesgue's convergence theorems in $R^1(\mu, \mathbb{R})$ are hold.

3.- The "μ-integrability" and the integral defined by Dunford-Sch-Schwartz in [4] , are special cases of the "abstract-Riemann-integrability" of [5] ; and only if Ω is an algebra and $\mu(X) < +\infty$, these concepts coincide (convergence in μ-measure, locally-μ-convergence and $\bar{I}$ -convergence are equivalent).

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