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DUALITY PROPERTIES OF INJECTIVE MODULES

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A Morita duality between two rings R and T is always induced by a faithfully balanced bimodule ${}_RU_T$ such that ${}_RU$ and U_T are injective cogenerators. In [13], an asymmetrical generalization of Morita duality has been given by considering *"duality R-modules"*, i.e., bimodules ${}_RU_T$ such that ${}_RU$ is a finitely cogenerated linearly compact quasi-injective self-cogenerator and T is naturally isomorphic to End$({}_RU)$. It has been remarked in [13] that these modules can be regarded as "1/2-Morita duality modules", because ${}_RU_T$ is a Morita duality module if and only if it is a duality R-module and a (right) duality S-module. However, they are in some sense rather more than "1/2-Morita duality modules" for if ${}_RU$ is a module and T = End$({}_RU)$, then U_T can be an injective cogenerator without ${}_RU$ being finitely cogenerated nor linearly compact nor a self-cogenerator. Thus the question arises of giving necessary and sufficient conditions on ${}_RU$ for U_T to be an injective cogenerator.

We will attack this problem by looking first at the simpler one of determining when U_T is injective, i.e., when ${}_RU$ is a *counterinjective* module. There is a result due to Würfel [12] and Damiano [4] which will be helpful for this purpose, namely, U_T is FP-injective (i.e., $\mathrm{Ext}_T(F,U) = 0$ for every finitely presented right T-module F) if and only if ${}_RU$ cogenerates all cokernels of homomorphisms $U^m \longrightarrow U^n$. It is also well known that the linear compactness of ${}_RU$ is closely related to the injectivity of U_T (see [8], [10]). But it is easily seen that a counterinjective module is not necessarily linearly compact (in the discrete topology) and so we will make use of the following more general concept: A left R-module X will be called *U-linearly compact* when each finitely solvable system of congruences $x \equiv x_i (\mathrm{mod}\ X_i)$, with the X_i U-closed submodules of X (i.e., such that X/X_i is U-cogenerated), is solvable [6]. We then get the following characterization of counterinjective modules:

Theorem 1. A left R-module ${}_RU$ is counterinjective if and only if the following conditions hold:

i) Every cokernel of a homomorphism of the form $U^m \longrightarrow U^n$ is U-cogenerated.
ii) U is U-linearly compact.

Several results scattered in the literature can be recovered as easy corollaries of Theorem 1. For instance we mention [8, Coroll. 1, p. 119], [10, Coroll. 2, p. 342], and [9 Theorem 1]. As another application, we get the following characterization of rings with Morita duality which improves [6, Corollary 3].

Corollary 2. Let ${}_RU_T$ be a faithfully balanced bimodule such that ${}_RU$ is a cogenerator and the injective envelope of U_T is cogenerated by U_T. Then R has a left Morita duality.

The proof consists in using results of [3] and [7] to show that ${}_RU$ is counterinjective and then one can apply Theorem 1.

If we consider faithfully balanced bimodules ${}_RU_T$ that are injective and counterinjective, it is easy to see that they induce a Morita duality (in the sense of [1]) between the quotient categories of R-Mod and Mod-T modulo the localizing subcategories defined by ${}_RU$ and U_T. The following result gives an indication of how close is R to having a Morita duality in this case. Recall that an R-module X is said to have U-dominant dimension ≥ 2 (U-dom dim X ≥ 2) if there exists an exact sequence $0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_2$, in which X_1 and X_2 are direct products of copies of ${}_RU$.

Theorem 3. Let ${}_RU$ be an injective and counterinjective module. If every direct sum of copies of ${}_RU$ has U-dom dim≥2, then T is semiperfect. If U is faithfully balanced and the class of modules of U-dom dim≥2 is closed under direct unions, then R has a left Morita duality.

As a consequence we get the following extension of [5, Coroll. 10.14]. Recall that an R-module X is called Σ-injective (resp. Δ-injective) when it is injective and R satisfies the ascending (resp. descending) chain condition on annihilators of subsets of X.

Corollary 4. Let ${}_RU_T$ be a faithfully balanced bimodule such that ${}_RU$ is Σ-injective (resp. Δ-injective) and U_T is injective. Then R is a left noetherian (resp. artinian) ring with a left Morita duality.

Recall that a module U is quasi-injective if, for every submodule X of U, the canonical homomorphism $\mathrm{Hom}_R(U,U) \longrightarrow \mathrm{Hom}_R(X,U)$ is an epimorphism. Using Theorem 1 we can now characterize the quasi-inujective modules ${}_RU$ such that U is an injective cogenerator.

Theorem 5. Let ${}_RU$ be a quasi-injective module and $T = \mathrm{End}({}_RU)$.

Then the following conditions are equivalent:

i) U_T is an injective cogenerator

ii) ${}_RU$ satisfies the following conditions:

a) ${}_RU$ cogenerates all the cokernels of homomorphisms $U \longrightarrow U^n$.

b) ${}_RU$ is U-linearly compact.

c) The lattice of U-closed submodules of ${}_RU$ has the finite intersection property.

iii) Every cyclic right T-module and every U-cogenerated quotient of U are U-reflexive.

We remark that the artinian injective modules ${}_RU$ which cogenerate an exact torsion theory (see [11] for the definition) satisfy all the conditions in ii) of Theorem 5, so that in this case, U_T is not only injective as asserted in [2, Theorem 4.2] but is also a cogenerator.

The proofs of the foregoing results, except Corollary 2, will appear in [6].

References

1. R.R. Colby and K.R.Fuller, QF-3' rings and Morita duality, Tsukuba J. Math. 1 (1984), 183-188.

2. G.D. Crown and J.J. Hutchinson, On the exactness of the completion functor, Commun. Algebra 8 (1980), 1-12.

3. R.R. Colby, Tilting modules, dominant dimension and exactness of duality functors, Tsukuba J. Math. 12 (1988), 441.449.

4. R.F. Damiano, Modules coflat over their endomorphism rings, Houston J. Math. 9 (1983), 373-382.

5. C. Faith, *Injective Modules over Levitzki Rings*, Lecture Notes in Pure Appl. Math., vol. 72 (Marcel Dekker, New York, 1982).

6. J.L. Gómez Pardo, Counterinjective modules and duality, to appear in J. Pure Appl. Algebra.

7. J.L. Gómez Pardo and N. Rodríguez González, On QF-3 and QF-3' modules and rings, unpublished.

8. T. Onodera, Linearly compact modules and cogenerators, J. Fac. Sci. Hokkaido U. 22 (1972), 116-125.

9. F.L. Sandomierski, Some examples of right self-injective rigs which are not left self-injective, Proc. Amer. Math. Soc. 26 (1970), 244-245.

10. F.L. Sandomierski, Linearly compact modules and local Morita duality, in: R. Gordon, ed., Ring Theory (Academic Press, New York, 1972).

11. B. Stenström, Rings of Quotients (Springer-Verlag, Berlin, 1975).

12. T. Würfel, Über absolut reine Ringe, J. Reine Angew. Math. 262/263 (1973), 381-391.

13. J.M. Zelmanowitz and W. Jansen, Duality for module categories, to appear in J. Algebra.