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EXAMPLES OF TAUBERIAN OPERATORS[1]

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Tauberian operators appeared in a problem of summability [4] and were studied by Kalton and Wilansky [10] and other authors [1, 5, 7, 11, 12]. Recently they have received some attention because they form a broader class than that of isomorphims (into), but yet they preserve some isomorphic properties of Banach spaces [8, 9].

Let $T^* \in L(Y^*,X^*)$ and $J(X)$ denote the conjugate operator of a continuous linear operator $T \in L(X,Y)$ and the immersion of a Banach space X in the second dual X^{**}. T is <u>tauberian</u> when $T^{**-1}J(Y) = J(X)$.

Upper semi-Fredholm operators (operators with closed range and finite dimensional kernel) are trivial examples of tauberian operators since they are isomorphisms up to finite dimensional subspace.

The main source of non trivial examples of tauberian operators is the celebrated factorization of Davis et al. (DFJP factorization) [3].

In this paper we show that all the conjugates of even order of the second factor in the DFJP factorization are tauberian operators. Then, using a particular case of a construction of Bellenot [2], we obtain a Banach space and a tauberian operator T in this space such that the second conjugate T^{**} is not tauberian, answering a question of Kalton and Wilansky [10]. Finally we present a simple example showing that the class of tauberian operators is not always open (another example showing it can be found in [11]) and prove that tauberian operators with closed range do not belong to the boundary of this class.

<u>Proposition</u> 1 Let j be the second factor in the DFJP factorization. $j^{(2n)}$ is a tauberian operator for every $n = 0, 1, 2, \ldots$

For $T \in L(X,Y)$ let $T_q \in L(X^{**}/X, Y^{**}/Y)$ given by $T_q(F+X) = T^{**}F + Y$

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Proposition 2 [12; 6, 2.8] T is tauberian if and only if T_q is injective. Moreover we can identify $(T_q)^{**} = (T^{**})_q$.

Observation 3 From the above characterization it is clear that for X^{**}/X reflexive, we have $T \in L(X,Y)$ tauberian if and only if so is T^{**}.

Now we show that there are tauberian operators whose second conjugate is not tauberian. We use a construction of Bellenot [2].

We denote by e_n the unit vector basis of ℓ_1, X_n the subspace of ℓ_1 generated by $\{e_1, \dots e_n\}$, and

$$J(X_n) := \{ (x_n) : x_n \in X_n, \|x_n\|_1 \longrightarrow 0 \text{ and } \|(x_n)\|_J < \infty \}$$

$$2\|(x_n)\|_J^2 := \sup \{ (\sum_{i=1}^{k-1} \|x_{n_{i+1}} - x_{n_i}\|^2) + \|x_{n_k}\|^2 : n_1 < n_2 < \dots < n_k \}.$$

Theorem 4 (1) $(J(X_n) , \| \|_J)$ is a Banach space.

(2) $J(X_n)^{**} = \{ (x_n) : x_n \in X_n , \|(x_n)\|_J < \infty \}$.

(3) $J(X_n)^{**}/J(X_n)$ is isometric to ℓ_1.

The operator $T \in L(\ell_1)$ given by $T(x_n) := (x_n/n)$ is injective, but T^{**} is not injective. On the other hand, let us consider in $J(X_n)$ the operator defined by $S(x_n) = (Tx_n)$.

Proposition 5 (a) $S \in L(J(X_n))$ and $S_q \equiv T$ (up to an isometry).

(b) S is tauberian, but S^{**} is not tauberian.

Observation 6 Denote by WCo the class of all weakly compact operators. In [7] it is proved that an operator $T \in L(X,Y)$ is tauberian if and only if for every space Z and $A \in L(Z,X)$ we have that $TA \in \text{WCo} \Rightarrow A \in \text{WCo}$.

We can consider a dual definition: $T \in L(X,Y)$ is cotauberian if and only if for every space Z and $B \in L(Y,Z)$ we have $BT \in \text{WCo} \Rightarrow B \in \text{WCo}$.

We have that $T \in L(X,Y)$ is cotauberian if and only if T^* is tauberian [6,7]. Moreover T^* cotauberian implies T tauberian. However the above example shows that the remaining implication is not true.

In general, the class of tauberian operators is not open in $L(X,Y)$. This fact, observed by Tacon [11], can be seen easily as follows.

Let X be a non-reflexive Banach space, and consider the operator defined by

$$T : (x_k) \in \ell_2(X) \longrightarrow (k^{-1}x_k) \in \ell_2(X)$$

Since $\ell_2(X)^{**} \equiv \ell_2(X^{**})$, it is clear that T and all its conjugates of even order are tauberian. However by the operators T_n given by

$$T_n : (x_k) \in \ell_2(X) \longrightarrow (x_1, 2^{-1}x_2, \dots, n^{-1}x_n, 0, 0, \dots) \in \ell_2(X)$$

verify $\|T_n - T\| = 1/n$, but clearly T_n are not tauberian.

Next result shows that the condition $R(T)$ not closed is essential.

Proposition 7 Tauberian operators with closed range do not belong to the boundary of the class of tauberian operators.

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