EXTRACTA MATHEMATICAE 4, n.2, 117-119 (1989)

NON-ARCHIMEDEAN HILBERT SPACES AND ADJOINT OPERATORS

J. A. Alvarez-García

Facultad de Ciencias. Universidad de Santander.(Spain)

A.M.S. Clasification (1980) : 46P05

Introduction.-In this paper we introduce "n.a. Hilbert spaces" in which the norm stems from a certain hermitian sesquilinear form, what is a generalization of the Springer spaces. We study the orthogonality and characterize the Riesz-Frechet mapping range. The adjoint of an operator is defined. It allows us to introduce an operator C*-algebra that is maximal in a n.a. Hilbert space. A matricial representation is given for the C*-algebra in the space c_o.Essential differences respect to the usual, real or complex case, are seen.

K will denote a complete field with characteristic different from 2, not trivial valued n.a., and with an involution $\alpha\in K \to \bar{\alpha}\in K$ and E will denote a linear space over K.

1.- Non archimedean inner products

(1.1) Definition.- A non archimedean inner product in E is a hermitian and anisotropic sesquilinear form $\varphi(x,y) \equiv (x,y)$ defined in E such that the mapping $\|.\|:x\in E \longrightarrow \|x\|:=|(x,x)|^{1/2}\in\mathbb{R}$ is a non-archimedean norm. If besides $(E,\|.\|)$ is complete, then E is said to be a n.a. Hilbert space.

(1.2) Theorem.- If the involution of K is isometric and $|2|=1$, then every hermitian and anisotropic sesquilinear form in E is a n.a. inner product that verifies $|(x,y)| \le \|x\|\|y\|$ $\forall x,y\in E$.

(1.3) Definition.- Given a vector system $S=\{e_i | i\in I\} \subset E-\{0\}$, we shall say that S is a N-orthogonal system (NOS) if for all $x = \sum_{k=1}^{n}\alpha_k e_{i_k}$ with $\alpha_k\in K$, $i_k\in I$, it holds $\|x\| = \max_{1\le k\le n} \|\alpha_k e_{i_k}\|$. We shall say that S is an E-orthogonal system (EOS) if $(e_i,e_j)=0$ for all $i\neq j$.

(1.4) Theorem.- Let E be a n.a. inner product space over K such that $|2|=1$. Then every EOS is a NOS.

The following terminology was introduced in [2]: A subspace M of a n.a. inner product space E over K is called orthocomplemented if $M \oplus M^{\perp}= E$. If $M^{\perp}$ is a normic complement of M too, M is said well complemented.

(1.6) Definition.- We shall call Riesz-Frechet dual of the space E to the range E_{RF} of h.

(1.8) Theorem.- Let E be a n.a. inner product space over K , and let $f \in E' - \{0\}$. The following conditions are equivalent:

(i) $N(f)^{\perp} \neq \{0\}$

(ii) $f \in E_{RF}$

If besides $|2|=1$, the following is equivalent too :

(iii) N(f) is orthocomplemented (then, well complemented).

2.- The n.a. Hilbert space of sequences

(2.1) Proposition.- Let K be with an isometric involution, and $|2|=1$. Then

$$(\ , \ ): (a,b) \equiv ((\alpha_n),(\beta_m)) \in c_o(K) \times c_o(K) \longrightarrow (a,b) := \sum_{n=1}^{\infty} \alpha_n \overline{\beta}_n \in K$$

is a hermitian and non-degenerate sesquilinear form. If (,) is anisotropic, then $c_o(K)$ is a n.a. Hilbert space. That situation occurs if K is one of the formal series fields K_i , i=1,2,3 of [1].

(2.2) Theorem (of Riesz-Frechet for $c_o(K)$).- Let K be with a continuous (in particular if isometric) involution, such that $c_o(K)$ is a n.a. Hilbert space, and $f \in c_o(K)'$. Then, the following properties are equivalent:

(i) $f \in c_o(K)_{RF}$

(ii) $\lim_n f(e_n) = 0$, where $e_n = (\delta_{nk} | k \in N)$; i.e. $(f(e_n)) \in c_o(K)$.

Furthermore, $c_o(K)_{RF}$ is a proper subspace of $c_o(K)' = l^{\infty}$.

3.- n.a. C*-algebras of operators

Along this section, E, F and G will be n.a. Hilbert spaces over K ; h_X will denote the Riesz-Frechet mapping of the n.a. Hilbert space X. We shall also suppose that $|2|=1$ (in K).

(3.1) Adjoint operators

Let us consider the subset $\mathcal{L}_o(E,F) := \{T \in Op(E,F) \mid D(T)^{\perp} = \{0\}\}$, and the mapping $* : T \in \mathcal{L}_o(E,F) \longrightarrow T^* \in Op(E,F)$.

(3.2) Proposition.- Let $T \in \mathcal{L}_o(E,F)$. Then :

(i) $D(T^*) = (T' \circ h_F)^{-1}(E_{RF})$

(ii) $T^* = h_E^{-1} \circ T' \circ h_F |_{D(T^*)}$.

(3.4) Theorem.- Let $A := \{T \in L(E) \mid T'(E_{RF}) \subset E_{RF}\}$. Then:

(i) $A = \{T \in L(E) \mid D(T^*) = E\}$

(ii) A is a non-commutative unitary Banach algebra over K.

(iii) The map * is an involution on A.

(iv) A is a n.a. C*-algebra.

(v) A contains the orthogonal projections of the well complemented subspaces of E.

4.- The C^*-algebra of operators on the n.a. Hilbert space $c_o(K)$

Let K be now with an isometric involution, where $|2|=1$ and such that $c_o(K)$ is a n.a. Hilbert space with the inner product of the section 2.

(4.1) Lemma.- Let E be a n.a. inner product space over K , and $T\in L(E)$. The mapping

$$\varphi_T: (x,y)\in E\times E \longrightarrow \varphi_T(x,y):=(Tx,y)\in K$$

is a bounded sesquilinear form in E such that $\|\varphi_T\| = \|T\|$.

(4.2) Lemma.- Let $E=c_o(K)$ and $\varphi: E\times E \longrightarrow K$ a bounded sesquilinear form. Then there exists an operator $T\in L(E)$ such that $\varphi=\varphi_T$, where φ_T is the sesquilinear form of the lemma (4.1), if and only if for all $x\in E$ is $\lim_i \varphi(x,e_i) = 0$. In such a case, the operator T is unique.

(4.3) Theorem.- Let $T\in L(c_o(K))$ and $A=\{T\in L(c_o) \mid T^*\in L(c_o)\}$. Then $T\in A$ if and only if for all $y\in c_o(K)$, $\lim_i (Te_i,y)=0$. Besides $A\neq L(c_o(K))$.

(4.4) Matricial representation of $T\in L(c_o(K))$

Given $T\in L(c_o(K))$, T determines and is determinated by the infinite matrix $[\alpha_{ij}]$ according to the equation $Tx = x[\alpha_{ij}]$, where the i-th row of $[\alpha_{ij}]$ is the coordinate vector of Te_i.

(4.5) Theorem.- Let $[\alpha_{ij}]$ be an infinite matrix of elements in K. Then :

(i) $[\alpha_{ij}]$ defines an operator $T\in L(c_o(K))$ if and only if it holds

(i-1) $\lim_j \alpha_{ij} = 0$ for every $i\in N$.

(i-2) $\sup_{i,j\in N} |\alpha_{ij}| < \infty$.

(ii) $[\alpha_{ij}]$ defines an operator $T\in A$ if and only if it holds (i-1),(i-2) and besides $\lim_i \alpha_{ij}=0$ for every $j\in N$. In such a case, the adjoint operator T^* of T is represented by the adjoint matrix $[\overline{\alpha_{ji}}]$ of the matrix $[\alpha_{ij}]$.

(4.6) Theorem.- Let $A_1:=\{T\in L(c_o(K)) \mid \lim_i Te_i=0\}$. Then A_1 is a closed $*$-subalgebra of A without unity.

REFERENCES

[1] Alvarez García, J.A. : Involutions on non-archimedean fields and algebras. Actas XIII J. Hisp-Lusas de Mat. Valladolid (1988).

[2] Bayod Bayod, J.M. : Productos internos en espacios normados no arquimedianos. Doctoral dissertation, Universidad de Bilbao, (1976).

[3] Monna, A.F. : Analyse non-Archimedienne. Springer-Verlag, (1970).

[4] Narici, L./Beckenstein, E./Bachman, G. : Functional Analysis and valuation theory. Marcel Dekker, (1971).

[5] Rooij, A.C. Van : Nonarchimedean functional analysis. M. Dekker (1978).