EXTRACTA MATHEMATICAE 4, n.3, 139-141 (1989)

GAP AND PERTURBATION OF NON-SEMI-FREDHOLM OPERATORS

Jose A. Alvarez[x], Teresa Alvarez[+], Manuel Gonzalez[x]

[x]Departamento de Matematicas, Universidad de Cantabria, Spain

[+]Departamento de Matematicas, Universidad de Oviedo, Spain

A.M.S. Classification (1980): 47A55

A classic result of Fredholm theory states that perturbing a semi-Fredholm operator T (i.e., an operator with closed range R(T) and finite dimensional kernel N(T) or cokernel Y/R(T)) by a continuous operator A with norm smaller than the minimum modulus $\gamma(T)$ of T (or by a compact operator A) we obtain a semi-Fredholm operator T+A; moreover dim N(T+A) ≤ dim N(T) and codim R(T+A) ≤ codim R(T) (see [3]).

Summing up, for operators with closed range the finite dimension of the kernel and the cokernel are stable under small or compact perturbations.

However, for any operator with non-closed range we can find a compact operator with arbitrarily small norm such that the perturbed operator has infinite dimensional kernel and cokernel [4].

In [1] Bouldin divides the class NSF of all bounded non-semi-Fredholm operators in a separable Hilbert space, in six disjoint sets U_i $0 \le i \le 5$. He proves that for $0 \le j \le 4$ the sets are totally unstable: U_j + Co = NSF. However the class U_5 of all operators with closed range and infinite dimensional kernel and cokernel is not totally unstable.

Later, Gonzalez and Onieva [5] studied the problem for bounded operators in Banach spaces, obtaining the same results for U_j $0 \le j \le 4$. These results were extended by Labuschagne [7] to closed operators. However in [5] and [7] the stability of U_5 is not investigated.

In this paper we shall prove that certain isomorphic properties of Banach spaces (separability, reflexivity, containing no copies of ℓ_1 ,...) for the kernel or the cokernel of a closed operator with closed range are stable under small or compact perturbations. As an application we prove that some subsets of U_5 are not totally unstable under compact perturbation.

Supported in part by DGICYT grant PB88-0417.

In order to do that, we use several known results about the stability of some properties of the subspaces of a Banach space under small perturbations measured by the gap (see [6, 9]). We also obtain here some new results of this kind and give new proofs for some of the known ones. For example, it will be shown that given a property P which is stable under the gap, the property P^{co} ($X \in P^{co} \Leftrightarrow X^{**}/J(X) \in P$) is also stable. From here we derive the stability of the reflexivivity [2; Th. 5] and the quasireflexivity as a consequence of the stability of the finite dimension.

Given two closed subspaces M, N of X, M° and $\delta(M,N)$ will be the annihilator of M and the gap between M and N, defined by

$$\delta(M,N) := \sup \{ \text{dist}(m,N) : m \in M, \|m\| = 1 \}.$$

The minimum modulus is $\gamma(T) := \inf \{\|Tx\| : \text{dist}(x,N(T)) = 1 \}$

<u>Theorem</u> (Markus [8]) Let T be a closed operator with $\gamma(T) > 0$, A a continuous operator and $S = T+A$.

(a) $\delta[N(S),N(T)] \le \gamma(T)^{-1}\|S-T\|$.

(b) $\delta[R(T),R(S)] \le \gamma(T)^{-1}\|S-T\|$.

<u>Definition</u> 1 A property P will be a class of Banach spaces stable under isomorphisms. A Banach space X has property P when $X \in P$.

Let $0 < a \le 1$. P is <u>a-stable</u> (<u>a-costable</u>) if for every Banach space X and closed subspaces M, N of X we have:

$N \in P$, $\delta(M,N) < a \Longrightarrow M \in P$. ($X/M \in P$, $\delta(M,N) < a \Longrightarrow X/N \in P$).

P is <u>stable</u> (<u>costable</u>) if it is a-stable (a-costable) for some a.

<u>Observation</u> 2 (a) The dual property $P^{d} := \{ X : \text{the dual } X^* \in P \}$ is a-costable (a-stable) when the property P is a-stable (a-costable).

(b) Finite dimension is a 1-stable and 1-costable property.

<u>Theorem</u> 3 (a) The classes of superreflexive, containing no copies of ℓ_1 and B-convex Banach spaces are 1-stable.

(b) The class of superreflexive Banach spaces is 1-costable.

(c) The class of separable spaces is 1/2-stable and 1/2-costable.

<u>Theorem</u> 4 If P is an a-stable (a-costable) property then the property $P^{co} := \{ X : H(X) \in P \}$ is a/2-stable (a/2-costable).

In particular, the quasireflexivity and the reflexivity are 1/2-stable and 1/2-costable; and { X : H(X) is separable}, studied in [10] is 1/4-stable and 1/4-costable.

Proposition 5 (a) If P is a-costable then P is a/2-stable.

(b) If P is a-stable then P is a/2-costable.

(c) The classes of superreflexive, containing no copies of ℓ_1 and B-convex Banach spaces are 1/2-costable.

Theorem 6 Let P be a property, T a closed operator with $\gamma(T) > 0$, $0 < a \leq 1$ and A a continuous operator such that $\|A\| < a\gamma(T)$.

(a) If P is a-stable and $N(T) \in P$, then $N(T+A) \in P$.

(b) If P is a-costable and $Y/R(T) \in P$, then $Y/\overline{R(T+A)} \in P$.

Theorem 7 Suppose $X \times N \in P \Leftrightarrow X \in P$, when $\dim N < \infty$. Let T be a closed operator with $\gamma(T) > 0$, K a compact operator, and $0 < a \leq 1$.

(a) If P is a-stable and $N(T) \in P$, then $N(T+K) \in P$.

(b) If P is a-costable and $Y/R(T) \in P$, then $Y/\overline{R(T+K)} \in P$.

Proposition 8 Let P be a property such that $X \times Y \in P$ when $X, Y \in P$. If P verifies the result in 8.a or 8.b, then P has the three space property; i.e., $X \in P$ when it has a subspace M such that $M, X/M \in P$.

REFERENCES

[1] R. Bouldin: "The instability of non-semi-Fredholm operators under compact perturbations". J. Math. Anal. Appl. 87 (1982), 632-638.

[2] A.L. Brown: "On the space of subspaces of a Banach space". J. London Math. Soc. 5 (1972), 67-73.

[3] S. Goldberg: "Unbounded linear operators". McGraw-Hill, 1966.

[4] M.A. Goldman: "On the stability of the property of normal solvability of linear equations (russian)". Dokl. Akad. Nauk. SSSR 100 (1955), 201-204.

[5] M. Gonzalez, V.M. Onieva: "On the instability of non-semi-Fredholm operators under compact perturbations". J. Math. Anal. Appl. 114 (1986), 450-457.

[6] R. Janz: "Perturbation of Banach spaces". Preprint Universitt Konstaz, 1987.

[7] L.E. Labuschagne: "On the instability of non-semi-Fredholm closed operators under compact perturbations with applications to ordinary differential operators". Proc. Edinburgh Math. Soc. 109A (1988), 97-108.

[8] A.S. Markus: "On some properties of linear operators connected with the notion of gap (russian)". Kishinev Gos. Univ. Uchen. Zap. 39 (1959), 265-272.

[9] M.I. Ostrowski: "On properties of the opening and related closeness characterizations of Banach spaces". Amer. Math. Soc. Transl. 136 (1987), 109-119.

[10] M. Valdivia: "On a class of Banach spaces". Studia Math. 60 (1977), 11-13.