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ON THE INDEX OF WEAKLY FREDHOLM OPERATORS

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A bounded linear operator $T : X \longrightarrow Y$ between normed spaces is called weakly Fredholm if its kernel $N(T)$ and its cokernel $Y/\overline{R(T)}$ are finite-dimensional.

T is Fredholm if it is weakly Fredholm and induces an invertible operator from $X/N(T)$ onto $\overline{R(T)}$.

For a weakly Fredholm operator T we may define its index by

$$\mathrm{ind}(T) := \dim N(T) - \dim Y/\overline{R(T)}$$

If $T : X \longrightarrow Y$ and $S : Y \longrightarrow Z$ are weakly Fredholm operators, following [2] we shall say that the pair (S,T) has the index property when the equality $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$ holds.

Much of the interest in the index lies in the fact that a pair of Fredhom operators has the index property [3; Theorem 6.5.4]. This result has been extended to operators with generalized inverses in [4].

Given a pair (S,T) of weakly Fredholm operators, ST is weakly Fredholm. However we cannot assure that (S,T) has the index property [2].

In this paper we obtain a characterisation of pairs (S,T) having the index property, from which we derive the sufficient conditions given in [2] and some additional ones.

In section 2, given two (closed) subspaces $M \subset X$ and $N \subset Y$ such that $TM \subset N$, we consider the decomposition of T into operators T_s and T_q defined by $T_s x := Tx \ (x \in M)$ and $T_q(x+M) := Tx+N \ (x \in X)$.

We prove that T is weakly Fredholm when T_s and T_q are so; however the equality $i(T) = i(T_s) + i(T_q)$ is not always true. We characterise the decompositions for which the equality holds.

If M is a subspace of the conjugate X^* of a normed space X, we denote by $\overline{M}^{w^*}$ the closure of M with respect to the topology $\sigma(X^*,X)$. Moreover $M \simeq N$ will mean that M and N are algebraically isomorphic.

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1. The index of a product

1.1 Proposition Let X, Y be normed spaces and let T^* be the conjugate operator of $T : X \longrightarrow Y$. We have

(a) $\dim Y/\overline{R(T)} = \dim N(T^*)$.

(b) $\dim N(T) = \dim X^*/\overline{R(T^*)}^{w^*} \le \dim X^*/\overline{R(T^*)}$.

(c) T^* weakly Fredholm $\Rightarrow$ T weakly Fredholm and $i(T^*) \le i(T)$.

1.2 Theorem. A pair (S,T) of weakly Fredholm operators has the index property if and only if

$$\dim \overline{R(T)}\cap N(S)/R(T)\cap N(S) = \dim N(T^*)\cap\overline{R(S^*)}^{w^*}/N(T^*)\cap R(S^*).$$

The index property for (S,T) depends more on the relation between S and T than on the individual properties of the operators. However we can derive sufficient conditions for the index property, in terms of S and T.

1.3 Corollary. A pair (S,T) of weakly Fredholm operators has the index property in the following cases:

(a) $R(T)$ is closed and $\overline{R(S^*)}^{w^*} = R(S^*)$.

(b) S is one-one and $R(T)$ is dense.

(c) S is bounded below.

(d) T is onto.

(e) The space Y is reflexive and $R(T)$, $R(S^*)$ are closed.

If, in addition, $X = Y$ and the pair commutes, then we also have

(f) T is one-one and $R(T)$ is dense.

(g) S is Fredholm of finite ascent and descent.

1.4 Observation Conditions (b), (c), (d), (f) and (g) were obtained in [2] by a different method. We observe that for these conditions we have always

$$\dim \overline{R(T)}\cap N(S)/R(T)\cap N(S) = \dim N(T^*)\cap\overline{R(S^*)}^{w^*}/N(T^*)\cap R(S^*) = 0.$$

Below we show an example of pair (S,T) which has the index property but it verifies $\dim \overline{R(T)}\cap N(S)/R(T)\cap N(S) \ne 0$.

1.5 Example Let $B \in L(\ell_2)$ given by $B(x_n) := (x_n/n)$. We have that $e := (1/n) \in \overline{R(B)}\setminus R(B)$. Then, denoting $f := e/\|e\|$, we define $A \in L(\ell_2)$ by $A(x) := x - f(x)f$ $x \in \ell_2$. Note that $\ell_2^* = \ell_2$.

A and B are weakly Fredholm operators, and $\text{ind}(A) = \text{ind}(B) = 0$. Moreover $\text{ind}(BA) = 1 = -\text{ind}(AB)$, since $N(AB) = \{0\}$, $N(BA) = N(A) \ne \{0\}$,

$\overline{R(AB)} = R(A) \neq \ell_2$, $\overline{R(BA)} = \ell_2$. Then (A,B), (B,A) do not have the index property. If

$$(S, T) = \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \right], \quad \text{we have} \quad ST = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix},$$

then $ind(S) = ind(T) = ind(A) + ind(B) = 0 = ind(AB) + ind(BA) = ind(ST)$. Hence (S,T) has the index property. However it is

$$\overline{R(T)} \cap N(S) = \overline{R(B)} \cap N(A) = N(A) \neq \{0\} = R(T) \cap N(S) = R(B) \cap N(A).$$

2. Decompositions of an operator

Given $T : X \longrightarrow Y$ and subspaces $M \subset X$, $N \subset Y$ with $TM \subset N$, we can "decompose" T in two operators $T_s \in L(M,N)$, $T_q \in L(X/M, Y/N)$ defined by

$$T_s m := Tm \quad (m \in M) \qquad \text{and} \qquad T_q(x+M) := Tx+M \quad (x \in X).$$

2.1 Proposition (a) $\dim N(T) \leq \dim N(T_s) + \dim N(T_q)$.

(b) $\dim Y/\overline{R(T)} \leq \dim N/\overline{R(T_s)} + \dim (Y/N)/\overline{R(T_q)}$.

If T_s and T_q are weakly Fredholm, then T is weakly Fredholm.

2.2 Example We shall show that, in general $ind(T)$ does not coincide with $ind(T_s) + ind(T_q)$. Let $X = Y = \ell_2 \oplus \ell_2$ and $M = N = \ell_2 \oplus \{0\}$.

Denoting by e_n the canonical basis of ℓ_2 we define $T \in L(\ell_2 \oplus \ell_2)$ by

$T(e_n,0) = (e_n/n,0)$ $\quad T(0,e_{n+1}) = (0,e_n)$ for $n = 1,2,3,\ldots$ and

$$T(0,e_1) = \sum_{n=1}^{\infty}(1/n)(e_n,0) \in \overline{R(T_s)} \backslash R(T_s).$$

We have $i(T) = 0 \neq i(T_s) + i(T_q) = -1$.

2.3 Theorem. Suppose T_s and T_q are weakly Fredholm. Then
$i(T) = i(T_s) + i(T_q)$ if and only if $\dim (R(T) \cap N)/TM = \dim (\overline{R(T)} \cap N)/\overline{TM}$.

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