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NONTRIVIAL SOLUTIONS TO NONLINEAR VOLTERRA EQUATIONS

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In some physic al problems the following nonlinear Volterra equations

$$u(x)=\int_0^x k(x-s)g(u(s))ds \tag{1}$$

where

(k) $k:(0,\delta_o) \longrightarrow (0,+\infty)$, $(\delta_o>0)$, is an absolutely continuous monotonous function such that $\int_0^{\delta_o} k(s)ds < +\infty$

and

(g) $g:|0,+\infty) \longrightarrow |0,+\infty)$ is an absolutely continuous increasing function such that $g(0)=0$ and $g(x)>0$ for $x>0$,

are considered (see |4|). With respect to physical applications the continuous solutions u to (1) such that $u(x)>0$ for $x>0$ are interesting. These are so-called nontrivial solutions. On the base of Gripenberg's work (see |3|) we can formulate:

Theorem 1. Let $\alpha>0$. Assume:

i) $g(u)/u$ is continuous positive decreasing such that $g(u)/u \longrightarrow +\infty$ as $u \longrightarrow 0+$.

ii) For each $q>0$ the function $u[g(u)/u]^q$ is increasing on $[0,\delta_q]$, $(\delta_q>0)$. The equation

$$u(x)=\int_0^x (x-s)^{\alpha-1}g(u(s))ds \tag{2}$$

has a nontrivial solution on $[0,\delta]$, $(\delta>0)$ if and only if

$$\int_0^{\delta} [u/g(u)]^{1/\alpha} \frac{du}{u}<+\infty \tag{3}$$

But we must emphasize that under Gripenberg's assumptions the case $g(u)=u^p$ $(p\in(0,1))$ is not allowed. In papers |1|, |2| and |5| under weaker assumptions of g similar results to Theorem 1 are presented. At these works the case $g(u)=u^p$ $(p\in(0,1))$ is allowed. Here we wat to present generalizations of condition (3). In all Theorems bellow K-1 will denote the inverse function to $K(x)=\int_0^x k(s)ds$. Now we formulate two theorems with sufficient conditions for the existence of nontrivial solutions to (1).

Theorem 2. Let k be an increasing function satisfying (k) and g satisfy (g). If

$$\int_0^{\delta_0} \frac{g'(u)}{g(u)} K^{-1}(u/g(u))du < +\infty \tag{4}$$

then equation (1) has a nontrivial solution on $[0,\delta]$, $(\delta>0)$.

Corollary 1. If g satisfies additionally (i) then $ug'(u)\leq g(u)$. Suppose

$$\int_0^{\delta_0} K^{-1}(u/g(u)) \frac{du}{u} < +\infty \tag{5}$$

Then (4) is satisfied and by Theorem 2 equation (1) has a nontrivial solution.

Theorem 3. Let k be a decreasing function satisfying (k) such that log k is convex and g satisfies (g). If

$$\int_0^{\delta_0} [g(u)k \circ K^{-1}(u/g(u))]^{-1} du < +\infty \tag{6}$$

Then equation (1) has a nontrivial solution on $[0,\delta]$, $(\delta>0)$.

Now we give two theorems concerning necessary conditions.

Theorem 4. Let k be an increasing function satisfying (k) such that log k is concave and g satisfies (g). If equation (1) has a nontrivial solution then

$$\int_0^{\delta_0} [g(u)\ k \circ K^{-1}(u/g(u))]^{-1} du < +\infty \tag{7}$$

Theorem 5. Let k be a decreasing function satisfying (k) and g satisfying (g). If equation (1) has a nontrivial solution then

$$\int_0^{\delta_0} [g'(u)/g(u)] K^{-1}(u/g(u))du < +\infty \tag{8}$$

Corollary 2. If there exists $q_0>1$ such that $u[g(u)/u]^{q_0}$ is increasing then $g'(u)\geq(1-1/q_0)\ g(u)/u$ (|3|). In this case if equation (1) has a nontrivial solution then

$$\int_0^{\delta_0} K^{-1}(u/g(u)) \frac{du}{u} < +\infty \tag{9}$$

Remark 1. Let us note that in the case of $k(x)=x^{\alpha-1}$ the conditions (5), (6), (7) and (9) are equivalent to Gripenberg's condition (3). By Corollary 1, Theorems 3 and 4 and Corollary 2 we get Theorem 1. Moreover all these conditions work in the case of $g(u)=u^p$ $(p\in(0,1))$.

Remark 2. It is known that equation (1) has a nontrivial solution in the case of $k(x)=\exp(-1/x^{\rho})$ $(\rho\geqslant 1)$ and $g(u)=u^p(p\in(0,1))$ But our sufficient conditions do not work in this case.

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