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APROXIMATION OF CONVEX BODIES BY POLYNOMIAL BODIES I: EXISTENCE THEOREMS

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There are several isolated results, without a common embodyment, about the approximation of convex bodies by means of polynomial bodies. We mention the following: For the density Hammer (1963) [2], proves that every convex body is the limit of polynomial bodies, and every symmetric convex body is the limit of bodies of positive definite homogeneous polynomials. As far as the author knows, the only result about existence, unicity, and characterization, refers to homogeneous polynomials of degree 2, and it is due to *Loewner* (see *Day* (1947) [1]). This result states that given a symmetric convex body in the plane, there exist unique maximum and minimum area ellipses, inscribed and circumscribed respectively to the sphere of the convex, and they touch the sphere in at least 4 points.

In this paper we give answer to the problems of existence of the best approximation of B by means of homogeneous polynomial bodies for three measures of deviation -area, width and radius-, and two types of approximation -interior and exterior-.

1. We consider in $\mathbb{R}^2$ a norm $\| \ \|$, with unit ball B and sphere S.

Let $\mathcal{P}_{2k}(\mathbb{R}^2,\mathbb{R})$ denote the (2k+1)-dimensional linear space of all homogeneous polynomials of degree 2k in $\mathbb{R}^2$, endowed with the sup-norm.

We say that $P\in\mathcal{P}_{2k}(\mathbb{R}^2,\mathbb{R})$ is *positive semidefinite* (*positive definite*), if $P(x)\geq 0$ (>0) for all $x\in\mathbb{R}^2-\{0\}$. Let $\mathcal{P}_{2k}$ denote the subset of $\mathcal{P}_{2k}(\mathbb{R}^2,\mathbb{R})$ of all positive semidefinite homogeneous polynomials.

For each $P\in\mathcal{P}_{2k}$ we consider the sets $B_P=\{x\in\mathbb{R}^2:P(x)\leq 1\}$ and $S_P=\{x\in\mathbb{R}^2:P(x)=1\}$, that we call the *polynomial body* and the *polynomial sphere* of **P** respectively. We define the *radius function of the unit sphere* S, like $r(\theta)=\|(\cos\theta,\sin\theta)\|^{-1}$. Likewise, we define the *radius function of a polynomial sphere* S_P, as the function $r_P(\theta)=[P(\cos\theta,\sin\theta)]^{-1/2k}$, (with $r_P(\theta)=+\infty$ if $P(\cos\theta,\sin\theta)=0$). We denote by m the Lebesgue measure, regardless of the dimension of the space.

2. Let $P\in\mathcal{P}_{2k}$, we say that B_P is *exterior* to B if $B\subset B_P$ ($r\leq r_P$), and we define $\mathcal{P}^e_{2k}(B)=\{P\in\mathcal{P}_{2k}\colon B\subset B_P\}$ -if we have no confusion with the degree 2k of the polynomials, we only write $\mathcal{P}^e(B)$–. The set $\mathcal{P}^e(B)$ is a nonempty, convex and closed subset of the unit ball of the linear space $\mathcal{P}_{2k}(\mathbb{R}^2,\mathbb{R})$.

We say that B_P is *interior* to B if $B_P \subset B$ ($r_P \leq r$). We denote by $\mathcal{P}^i_{2k}(B)$ (or $\mathcal{P}^i(B)$ if there is no confusion) this set, that it is a nonempty, convex, closed and unbounded subset of the set $Int[\mathcal{P}_{2k}]$, of all positive definite homogeneous polynomials of degree 2k,.

3.- Let $Q \in \mathcal{P}^e(B)$. We say that B_Q is an *exterior-area best approximation* of B if $m[B_Q]=\inf\{m[B_P]:B \subset B_P\}$. We denote by $\mathcal{B}^e_a(B)$ the set of all such best approximations. Let $Q \in \mathcal{P}^i(B)$. We say that B_Q (or S_Q or **Q**) is an *interior-area best approximation* of B if $m[B_Q]=\sup\{m[B_P]:P \in \mathcal{P}^i(B)\}$. We denote by $\mathcal{B}^i_a(B)$ the set of all such best approximations. For each $P \in \mathcal{P}_{2k}$ we define the *width of* B_P, to be the ratio between the radius of the circumscribed ball and the radius of the inscribed ball to B_P, that is to say $w(P)=\sup\{\|x\|:x \in S_P\}/\inf\{\|x\|:x \in S_P\}$. We say that B_Q is an *exterior-width best approximation of* B if it satisfies $B \subset B_Q \subset w(Q)B$ and $w(Q)=\inf\{w(P):P \in \mathcal{P}^e(B)\}$. We denote by $\mathcal{B}^e_w(B)$ the set of all such best approximations. Let $Q \in \mathcal{P}^i(B)$. We say that B_Q (S_Q or **Q**) is an *interior-width best approximation* of B if $w(Q)=$ $=\inf\{w(P):P \in \mathcal{P}^i(B)\}$ and $w^{-1}(Q)B \subset B_Q \subset B$. We denote by $\mathcal{B}^i_w(B)$ the set of all such best approximations. Let $Q \in \mathcal{P}^e(B)$. We say that B_Q is a *radius-exterior best approximation* of B if $\|r_Q-r\|_\infty=\inf\{\|r_P-r\|_\infty:P \in \mathcal{P}^e(B)\}$. We denote by $\mathcal{B}^e_r(B)$ the set of all such best approximations. Let $Q \in \mathcal{P}^i(B)$, we say that B_Q is an *interior-radius best approximation* of B if $\|r_Q-r\|_\infty=\inf\{\|r_P-r\|_\infty:P \in \mathcal{P}^i(B)\}$. We denote by $\mathcal{B}^i_r(B)$ the set of all such best approximations.

4. It is obvious that if the optimum with the width-exterior criterion exists, then its polynomial will be in the compact set $K_W=\{P \in \mathcal{P}^e(B)$: $B \subset B_P \subset$ $\subset (R/r)B\}$, where $R=\max r$, $r=\min r$. And on it the function $D_W(P)=\max[r_P/r]$ is continuous, because the convergence $P_n \to P$ implies the uniform convergence $r_{P_n} \to r_P$. If the optimum with the radius-exterior criterion exists its polynomial must be in the compact set $K_r=\{P \in \mathcal{P}^e(B): \|r_P-r\|_\infty \leq R-r\}$ and on it the function $D_r(P)=\max[r_P-r]$ is continuous. Then we have

THEOREM (1). The sets $\mathcal{B}^e_w(B)$ and $\mathcal{B}^e_r(B)$ are nonempty.

In the area case the situation becomes slightly more complex, because the continuity of $D_a(P)=m[B_P]$ on a suitable compact set is not clear. However, in order to prove the existence of an exterior-area best approximation of B, we only need to show the lower semicontinuity of such function

D_a on the compact set $\mathcal{P}^e(B)$. As a consequence of that we have

THEOREM (2). The set $\mathcal{B}_a^e(B)$ is nonempty.

5.- For the interior approximation we have that $B_P \in \mathcal{B}_w^1(B)$ iff $w(P)B_P \in \mathcal{B}_w^e(B)$, then we have

THEOREM (3). The set $\mathcal{B}_w^1(B)$ is nonempty.

Nevertheless, for the other two criteria the situation is more difficult because $\mathcal{P}^1(B)$ is not compact. Then we will try another way that will give us some information to understand the behaviour of a polynomial body B_P as a point of its sphere gets closer to the origin.

THEOREM (4). Let $P_n \in \mathcal{P}^1(B)$ be a sequence of polynomials with radius functions r_n, such that $r_n(0) \to 0$. Then $m[B_{P_n}] \to 0$.

Now using **(4)** and the continuity of $D_a(P)=m[B_P]$ on $Int[\mathcal{P}_{2k}]$ -and then on $\mathcal{P}^1(B)$-, we have the

THEOREM (5). The set $\mathcal{B}_a^1(B)$ is nonempty.

Although from a geometric standpoint the radius and width criteria are very similar, there are some important differences between them. One of such differences appears in the proof of the existence theorem of best approximation. A similar result to **(4)** give us a way to prove the

THEOREM (6). The set $\mathcal{B}_r^1(B)$ is nonempty.

REFERENCES

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[2] *HAMMER,P.C. :"Approximation of convex surfaces by algebraic surfaces". Mathematika, 10, 64-71, 1963.*

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