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NEW CONDITIONS FOR THE EXISTENCE OF NONTRIVIAL SOLUTIONS TO SOME VOLTERRA EQUATIONS

by

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We consider the following Volterra equation:

$$u(x) = \int_0^x k(x-s)g(u(s))ds \tag{1}$$

where:

$k:[0,\delta_0] \longrightarrow \mathbb{R}$ is an increasing absolutely continuous function such that $k(0)=0$ (k)

$g:[0,+\infty) \longrightarrow [0,+\infty)$ is an increasing absolutely continuous function such that $g(0)=0$ and $g(u)/u \longrightarrow \infty$ as $u \longrightarrow 0^+$ (see [3]) (g)

Let us note that (1) has always the trivial solution u=0.

Some necessary and sufficient conditions for the existence of nontrivial solutions to (1) with $k(x)=x^{\alpha-1}$ ($\alpha>0$) are given in [1],[2] and [4]. In [3] and [5] conditions for more general kernels are presented. But those results do not give answers about nontrivial solutions in the case of kernels which are very smooth near the origin. Now we are able to show the following necessary condition:

Theorem 1. Assume (k) and (g) are satisfied. If equatio (1) has a nontrivial solution then

$$\sum_{n=0}^{\infty} K^{-1}\left[(g^{-1})^n(x) \;/\; g((g^{-1})^n(x))\right] < +\infty \tag{2}$$

for $x\in[0,\delta]$, ($\delta>0$).

In the formula above K^{-1} denotes the inverse function of $K(x)=\int_0^x k(s)ds$ g^{-1} is the inverse function of g , and $(g^{-1})^n$ is the nth iteration of g^{-1}

Moreover, the following sufficient condition is true:

Theorem 2. Assume (k) and (g) are satisfied. Let φ be an absolutely continuous function such that $u<\varphi(u)<g(u)$ on $(0,\delta]$, $(\delta>0)$, and $\varphi(u)/u \longrightarrow \infty$ as $u \longrightarrow 0^+$. If

$$\sum_{n=0}^{\infty} K^{-1}\left[(g^{-1}\circ\varphi)^n(x)/\varphi((g^{-1}\circ\varphi)^n(x))\right] < +\infty \tag{3}$$

on $[0,\delta]$, then equation (1) has a nontrivial solution.

Let us consider

$$u(x) = \int_0^x k(x-s)[u(s)]^p ds \quad ,p\in(0,1) \tag{4}$$

with $k=K'$, where $K(x)= \exp(-\exp 1/x^{\alpha})$,$\alpha>0$.

Applying Theorem 1 we obtain that (4) has not nontrivial solutions for $\alpha\geq1$. Putting $\varphi(u)=u^q$,where $q\in(0,1)$ and $q>p$, into (2) we get that the series is convergent for $\alpha\in(0,1)$. This means that (4) has a nontrivial solution for $\alpha\in(0,1)$.

It is shown in [5] that under additional assumptions the condition $\int_0^{\delta} K^{-1}(s/g(s))\frac{ds}{s}<+\infty$ implies the existence of nontrivial solutions. Since $s^{-1}K^{-1}(s/g(s))$ is decreasing then we have the following inequality:

$$\sum_{n=0}^{\infty} K^{-1}\left[\frac{x}{2^n\, g(x/2^{n+1})}\right]\leq\int_0^{x/2} K^{-1}(s/g(s))\frac{ds}{s} \tag{5}$$

We can get the left-hand side of (5) putting $\varphi(u)=g(\frac{1}{2}u)$ into (3). This means tha (3) is a generalization of results presented in [6].

References

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