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SOME PROPERTIES OF THE HAUSDORFF DISTANCE IN METRIC SPACES

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ABSTRACT

Some properties of the Hausdorff distance in complete metric spaces are discussed. Results obtained in this paper explain ideas used in the theory of measures of noncompactness.

1. INTRODUCTION

The aim of this paper is to present some properties of the Hausdorff distance in complete metric spaces which are especially useful in the theory of measures of noncompactness.

Our considerations are closely related to measures of noncompactness defined in an axiomatic way (cf. [1], [2], [3], [4], [6], [7]), for example). More precisely, we are interested in the following problem: Let $\mathcal{Z}$ be a subfamily of the family $\mathcal{M}$ of all nonempty and bounded subsets of a metric space. For $X \in \mathcal{M}$ define the number $H_{\mathcal{Z}}(X)$ as the Hausdorff distance of X from $\mathcal{Z}$. What can we say about properties of the function $H_{\mathcal{Z}}$ if we assume that the family $\mathcal{Z}$ satisfies some ordered or topological conditions?

Our results obtained here explain some ideas used in axiomatic definitions of measures of noncompactness proposed up to now. Particularly, we provide short proof of a few theorems from the book [2] which are formulated here in a more general setting.

2. NOTATION

Let (M,ρ) be a complete metric space. We use the following notation:

$\mathcal{M} = \{X \subset M : X \text{ nonempty and bounded}\}$;

$\mathcal{N} = \{X \subset M : X \text{ nonempty and relatively compact}\}$;

$K(x,r) = \{y\in M : \rho(x,y)<r\}$, if $x\in M$ and $r>0$;

$K(X,r) = \bigcup_{x\in X} K(x,r)$, if $X\in\mathcal{M}$ and $r>0$;

$d(X,Y) = \inf \{r : X\subset K(Y,r)\}$, if $X,Y\in\mathcal{M}$;

$D(X,Y) = \max \{d(X,Y),d(Y,X)\}$, if $X,Y\in\mathcal{M}$;

$\overline{X}$ = the closure of a subset X of M .

The function D(X,Y) is called the Hausdorff distance between sets X and Y. is well known that D is a pseudometric on $\mathcal{M}$ and it is a complete metric on $\mathcal{A}$ Moreover, $\mathcal{N}^c$ forms a closed subspace of $\mathcal{M}^c$ with respect to the topolc generated by D [5].

If $\emptyset\neq\mathcal{Z}\subset\mathcal{M}$ then we will use the following notation:

$\mathcal{Z}^o = \{Z\in\mathcal{Z} : Z \text{ is finite}\}$;

$\mathcal{Z}^c = \{X\in\mathcal{Z} : X=\overline{X}\}$;

$D(X,\mathcal{Z}) = \inf \{D(X,Z) : Z\in\mathcal{Z}\}$;

$d(X,\mathcal{Z}) = \inf \{d(X,Z) : Z\in\mathcal{Z}\}$.

In what follows we shall consider the function $H_{\mathcal{Z}}: \mathcal{M} \longrightarrow [0,\infty)$ (or shor H) defined in the following way

$H_{\mathcal{Z}}(X)=D(X,\mathcal{Z})$,

where $\mathcal{Z}$ is the same as above.

3. MAIN RESULTS.

Theorem 1. Let $\mathcal{Z}$ be a nonempty subfamily of $\mathcal{M}$ satisfying the condition

(*) $X\in\mathcal{Z},\ \emptyset\neq Y\subset X \Rightarrow Y\in\mathcal{Z}$.

Then for any $X,Y\in\mathcal{M}$ the following propositions holds

a) $d(X,\mathcal{Z}) = D(X,\mathcal{Z})$.

b) $X\subset Y \Rightarrow H(X)\leq H(Y)$.

c) $\max \{H(X),H(Y)\} \leq H(X\cup Y)$

d) $X\cap Y\neq\emptyset \Rightarrow H(X\cap Y) \leq \min \{H(X),H(Y)\}$.

e) $H(X)=H(\overline{X})$

Theorem 2. If a family $\mathcal{Z}$ fulfills the condition (*) and the following o $A,B\in\mathcal{Z} \Rightarrow A\cup B\in\mathcal{Z}$, then $H(X\cup Y) = \max \{H(X),H(Y)\}$

Theorem 3. If $\mathcal{Z}$ is a family satisfying the condition (*) and $\mathcal{Z}\subset\mathcal{N}$, then

$H_{\mathcal{Z}}(X) = H_{\mathcal{Z}^o}(X) = d(X,\mathcal{Z}^o)$.

Theorem 4. $H_{\mathcal{Z}}(X)=0 \Leftrightarrow \overline{X}\in\overline{\mathcal{Z}}$ (the closure of $\mathcal{Z}$ in $\mathcal{M}^C$ with respect to the topology generated by D).

Theorem 5. Assume that $\mathcal{Z}$ satisfies the condition (*) and $\mathcal{Z}\subset\mathcal{N}$. Further, let $X_n\in\mathcal{M}^C$, $X_n\supset X_{n+1}$ for $n=1,2,\ldots$ and $\lim_{n\to\infty} H(X_n)=0$. Then the set $X_\infty=\bigcap_{n=1}^{\infty} X_n$ is nonempty and $X_\infty\in\overline{\mathcal{Z}}$.

4. EXAMPLES

Example 1. Let $\mathcal{Z}=\mathcal{N}$. Then the function H is called the Hausdorff measure of noncompactness in the space M. For its properties we refer to [2].

Example 2. Let $\mathcal{Z}$ be the family of all one-point sets in a metric space M. Obviously $\mathcal{Z}$ satisfies the condition (*) and $\mathcal{Z}\subset\mathcal{N}$. It is easy to show that in such a case the function $H_{\mathcal{Z}}$ may be expressed in the following way $H_{\mathcal{Z}}(X)=r(X)$, where

$$r(X) = \inf\left\{\sup\,\{\rho(x,y)\ :\ y\in X\}\ :\ x\in M\right\}$$

The number r(X) is called the radius of X.

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