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FINITE TOPOLOGICAL SPACES

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We show that the study of topological T_o-spaces with a finite number of points agrees essentially with the study of polyhedra, by means of the geometric realization of finite spaces. In this paper all topological spaces are assumed to be T_o.

If X is a partially ordered set, we may consider a topology (the *order topology)* on it by letting all hereditary subsets ($C \subseteq X$ such that $x \leq y \in C \Rightarrow x \in C$) be the closed sets. This topology determines the given order, since we have $x \leq y \Leftrightarrow$ x is in the closure of y. Conversely, any T_o-topology on a set X defines a partial order on X: $x \leq y \Leftrightarrow$ x belongs to the closure of y. The corresponding order topology is finer than the given topology and both topologies agree if and only if arbitrary unions of closed sets are also closed. Obviously, any finite topological space satisfies this condition. Moreover, a map f between finite spaces is continuous if and only if it is order preserving: $x \leq y \Rightarrow f(x) \leq f(y)$. Hence, *the category of finite spaces and continuous maps is isomorphic to the category of finite patially ordered sets and order-preserving maps.* We shall make no distinction between a finite space and its corresponding partially ordered set.

The *height* or *dimension* of a finite space is the supremum of all integers n such that there exists a chain $x_o < x_1 < \ldots < x_n$ in X.

Duality theorem: *Let* $f: X \longrightarrow Y$ *be a continuous map between finite spaces and let* $I^{\bullet}$ *be an injective resolution of the constant sheaf* $\mathbb{Z}$ *on* Y. *There exists a complex* $D_f^{\bullet}$ *of injectives sheaves on* X *such that, for every bounded complex* $K^{\bullet}$ *of abelian sheaves over* X *we have a functorial isomorphism*

$$\mathrm{RHom}^{\bullet}(\mathrm{R}f_*(K^{\bullet}), I^{\bullet}) = \mathrm{RHom}^{\bullet}(K^{\bullet}, D_f^{\bullet})$$

This complex $D_f^{\bullet}$ is well-defined up to quasi-isomorphisms, and we define the *dualizing complex* $D_X^{\bullet}$ of X to be $D_f^{\bullet}$ when Y is the one point space.

Theorem 1: *The homology groups of a finite space* X *are just the hypercohomology groups of its dualizing complex*

$$H_i(X,\mathbb{Z}) = \mathbb{H}^{-i}(X,D_X^{\bullet})$$

Geometric Realization of a finite space

If X is a finite space, we denote by βX the finite space of all chains $x_o<x_1<\dots x_n$ of points of X, where the closure of any such chain is the set of its subsequences. We say that βX is the *barycentric subdivision* of X.

If we assign to each chain its last element, we get a functorial continuous projection

$$s : \beta X \longrightarrow X$$

such that the inverse image of any closed subset Z of X is just βZ.

The *iterated barycentric subdivisions* $\beta^n X$ are defined inductively when $n \geq 2$

$$\beta^n X = \beta(\beta^{n-1}X)$$

so that we obtain an inverse system of finite spaces

$$\dots \longrightarrow \beta^{n+1}X \longrightarrow \beta^n X \longrightarrow \dots \longrightarrow \beta X \longrightarrow X$$

where $\beta^{n+1}X \longrightarrow \beta^n X$ is the projection of the barycentric subdivision of $\beta^n X$ onto $\beta^n X$.

Definition: We define the *geometric realization* $|X|$ of a finite space X to be the subspace of all closed points of the inverse limit $\varprojlim \beta^n X$.

By definition, we have a functorial continuous map $g: |X| \longrightarrow X$.

Given a finite space X, we may consider a simplicial complex $\mathcal{C}(X)$ on the vertex set X by letting all chains be the faces. Then, $|X|$ is the geometric realization of $\mathcal{C}(X)$.

Any finite space X may be represented by a diagram of edges joining some points of X in such a way that the lower vertex of each ascending path is in the closure of the upper vertex, so that $|X|$ is the following polyhedron: the vertices of $|X|$ are the points of X and there is a simplex in $|X|$ with vertices $x_o,x_1,\dots,x_n$ if and only if there is an ascending path in X joining all these points. For example, the geometric realizations of the following finite spaces are a segment and a circle respectively:

Theorem 2: *If $\mathcal{F}$ is a sheaf of abelian groups over a finite space* X, *then the inverse image*

$$g^* : H^p(X,\mathcal{F}) \longrightarrow H^p(|X|, g^*\mathcal{F})$$

is an isomorphism for all $p \geq 0$.

Theorem 3: $D^\bullet_X$ *and* $g_*(D^\bullet_{|X|})$ *are quasi-isomorphic.*

Theorem 4: *If* X *is a finite space, then each covering space of* $|X|$ *is the geometric realization of an unique covering space of* X, *so that the morphism*

$$g_* : \pi_1(|X|,p) \longrightarrow \pi_1(X,g(p))$$

is an isomorphism for any $p \in |X|$.

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