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A COUNTEREXAMPLE IN SEMIMETRIC SPACES

by

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Let X be a topological space and "d" a real valued nonnegative symmetric function defined on XxX, such that $d(x,y)=0$ if and only if $x=y$. The pair (X,d) is said to be a *semimetric space* when the closure of every $A\subseteq X$ is given by $\bar{A}=\{x\in X:\ d(x,A)=0\}$, where $d(x,A)=\inf\{d(x,a).\ a\in A\}$.

A *filter* $\mathcal{F}$ on a set X is a nonempty collection of subsets of X satisfying: 1) $\emptyset\notin\mathcal{F}$. 2) $M,N\in\mathcal{F}$ implies $M\cap N\in\mathcal{F}$. 3) $M\in\mathcal{F}$, $M\subseteq N$ imply $N\in\mathcal{F}$.

A collection $\mathfrak{B}$ of subsets of X is a *filter-base* on X if the set $\mathcal{F}$ of subsets of X which contain some $B\in\mathfrak{B}$ is a filter on X. In this case we will also say that $\mathfrak{B}$ is a base for the filter $\mathcal{F}$. If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are filter-bases on X, it is said that $\mathfrak{B}_1$ is *finer* than $\mathfrak{B}_2$ when each $B_2\in\mathfrak{B}_2$ contains a $B_1\in\mathfrak{B}_1$.

The filter-base $\mathfrak{B}$ is said to be *convergent* to a point x of a topological space X when for each neighborhood of x, U, there is a $B\in\mathfrak{B}$ such that $B\subseteq U$. It is easy to see that if X is a semimetric space, the filter-base $\mathfrak{B}$ converges to x if and only if each ball of center x contains some $B\in\mathfrak{B}$. A filter-base $\mathfrak{B}$ in a semimetric space (X,d) is called a d-*Cauchy* filter-base when for each $\varepsilon>0$ there is some $B\in\mathfrak{B}$ having d-diameter $\delta(B)<\varepsilon$, where we have put $\delta(B)=\sup\{d(x,y):\ x,y\in B\}$.

If $\{x_n\}$ is a sequence in X, then it is obvious that the filter-base $\{A_k\}$, $A_k=\{x_k:n\geq k\}$ (*elementary filter-base*) is d-Cauchy (or convergent to x) if and only if $\{x_n\}$ is d-Cauchy (or convergent to x).

The semimetric space (X,d) is said to be *Cauchy-complete* if every d-Cauchy sequence converges, and it is said to be *complete* if every d-Cauchy filter converges. Other different definitions of completeness is semimetric spaces can be found in [3] and [5].

It is clear that "complete" implies "Cauchy complete". The question arises whether every Cauchy complete semimetric space is complete. The answer is known to be positive for T_2-spaces [4], and we give in this paper a finer condition. We also obtain an example of a semimetric space (X,d) which is Cauchy complete but not complete.

Proposition. *Let* (X,d) *be a Cauchy complete semimetric space where every Cauchy filter with countable filter-base admits a finer elementary filter-base which converges to, at most, a countable quantity of points. Then* (X,d) *is complete.*

EXAMPLE. *A Cauchy complete semimetric space which is not complete:*

Let X be the closed unit circle (without its center), and let $x=(|x|, \arg x)$, with $2\pi \leq \arg x < 4\pi$, be a polar representation of the points $x \in X$. Define on X the following distance function:

$d(x,x) = 0$	for every $x \in X$
$d(x,y) = 1-\min(\|x\|,\|y\|)$	if $\|x\|<1$, $x \neq y$
$d(x,y) = d(y,x) = 1-\|x\|$	if $\|x\|<1$, $\|y\|=1$, $\arg x \neq \arg y$
$d(x,y) = d(y,x) = \arg x$	if $\|x\|<1$, $\|y\|=1$, $\arg x = \arg y$
$d(x,y) = \max(\arg x, \arg y)$	if $\|x\|=1$, $\|y\|=1$, $x \neq y$

Remark. If we replace the condition $2\pi \leq \arg x < 4\pi$ into $0 \leq \arg x < 2\pi$, and we take $d(x,y)=d(y,x)= \max(\arg y, 1-|x|)$ when $|x|<1$, $|y|=1$ and $\arg x = \arg y$, then we obtain a complete semimetric space where every convergent sequence converges to an uncountable number of points. Therefore the hypothesis of the proposition is not necessary. On the other hand, it is easy to find Cauchy complete not Hausdorff semimetric spaces which satisfy the condition of the proposition.

The preceding EXAMPLE (and proposition) were formerly obtained in a different context than semimetric spaces: that of diameter spaces, introduced and studied by Montalvo in [6].

Definition. A diameter space is a pair (X,δ), where X is a nonempty set and $\delta:\mathcal{P}(X) \longrightarrow [0,+\infty]$ a map (diameter) satisfying:

(δ_1) $\delta(A) \geq 0$ for all $A \subseteq X$

(δ_2) The empty set and sets with a single point have diameter 0. No other set has diameter 0.

(δ_3) Finite sets have finite diameter.

(δ_4) If $A \subseteq B$ then $\Delta(A) \leq \delta(B)$

(δ_5) If $(A_i)_{i \in I}$ is a family of subsets of X totally ordered by inclusion, such that $\delta(A_i) \leq K$ for each $i \in I$, then $\delta(\bigcup_{i \in I} A_i) \leq K$.

If (X,d) is a semimetric space, it is easy to see that the application

$$\delta(A) = \sup \{d(x,y): x,y\in A\}$$

defines a diameter on X. However not every diameter can be obtained in this form (see [6].

Several different notions of completeness can be defined in abstract diameter spaces. Results relating those notions of completeness for the diameter space generated by a semimetric d can be translated into results about d. Our results in this paper correspond to some other more general statements about the associated diameter space, as can be seen in its full generality in [6].

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