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THE COMPACT WEAK TOPOLOGY ON A BANACH SPACE

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Throughout, E and F will denote Banach spaces. The bounded weak topology on a Banach space E, noted $bw(E)$ or simply bw, is defined as the finest topology that agrees with the weak topology on bounded sets. It is proved in [3] that $bw(E)$ is a locally convex topology if and only if E is reflexive.

In this paper we introduce the compact weak topology on a Banach space E, noted $kw(E)$ or simply kw, as the finest topology that agrees with the weak topology on weakly compact subsets. Equivalently, kw is the finest topology having the same convergent sequences as the weak topology. This topology appears in a natural manner in the study of a certain class of continuous mappings.

We prove that $kw(E)$ is a locally convex topology if and only if

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the space E is reflexive or has the Schur property. We denote ckw the finest locally convex topology contained in kw, and derive characterizations of Banach spaces not containing ℓ_1, and of other classes of Banach spaces, in terms of these topologies. It is also shown that $ckw(E)$ is the topology of uniform convergence on (L)-sets of the dual space E^*. As a consequence, we characterize Banach spaces with the reciprocal Dunford-Pettis property.

1. Proposition. *Let E_{kw} denote the topological space (E, kw). Then:*

(a) *E_{kw} is a k-space (i.e. every subset having closed intersection with each compact subset is closed);*

(b) *E_{kw} is a sequential space (i.e. a subset $A \subset E$ is kw-closed if and only if each weakly convergent sequence in A has its limit in A).*

2. Theorem. (a) *The topology $kw(E)$ coincides with the norm topology if and only if E has the Schur property.*

(b) *$kw(E)$ and $bw(E)$ coincide if and only if E does not contain ℓ_1.*

By a general result [1], the kw topology is semilinear. Furthermore, we have:

3. Theorem. *The following assertions are equivalent:*

(a) *E is reflexive or Schur;*

(b) *$kw(E)$ is a locally convex topology;*

(c) *$kw(E)$ is a vector topology.*

A subset $A \subset E^*$ is said to be an (L)-set if for any weakly null sequence $(x_n) \subset E$, one has $\lim_n \sup_A |\langle x_n, x^* \rangle| = 0$ [2].

4. Theorem. *ckw(E) is the topology of uniform convergence on (L)-sets of* E^*.

5. Corollary [2]. *E does not contain an isomorphic copy of* ℓ_1 *if and only if (L)-sets in* E^* *are relatively compact.*

If τ_L denotes the topology on E^{**} of uniform convergence on (L)-sets of E^*, we have:

6. Theorem. *The following assertions are equivalent:*

(a) *E has the reciprocal Dunford-Pettis property;*

(b) *the unit ball of E is* τ_L*-dense in the unit ball of* E^{**}

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