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NOTE ON MEASURES OF NONCOMPACTNESS IN BANACH SEQUENCE SPACES

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1. INTRODUCTION. The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The current state of this theory and its applications are presented in the books [1,4,11], for example.

The notion of a measure of weak noncompactness was introduced by De Blasi [8] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations (cf. [2,3,9,10,11], for instance).

In this note we summarize our papers [5,6]. We study measures of noncompactness and measures weak of noncompactness in some Banach sequence spaces.

2. NOTATION. Assume that E is a Banach space. The unit ball of E will be denoted by B_E . Moreover $Conv\ X$ denote the convex closure of a set X and $\|X\| = \{\|x\| : x \in X\}$. Finally, denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E. A function μ: $\mathcal{M}_E \longrightarrow \mathbb{R}_+=[0,\infty)$ will be called a measure of noncompactness in E if it satisfies the following conditions:

1^o $\mu(X)=0 \Leftrightarrow X$ is relatively compact

2^o $X\subset Y \Rightarrow \mu(X)\leq\mu(Y)$

3^o $\mu(Conv\ X)=\mu(X)$

4^o $\mu(X\cup Y)=max\{\mu(X),\mu(Y)\}$

5^o $\mu(X+Y)\leq\mu(X)+\mu(Y)$

6^o $\mu(cX)=|c|\mu(X),\ c\in\mathbb{R}$

It is said to be a measure of weak noncompactness [7] in E if it satisfies the conditions 2^o–6^o and

1^o $\mu(X) = 0 \iff X$ is weakly relatively compact

Recall that the functions χ, β: $\mathcal{M}_E \longrightarrow \mathbb{R}_+$ defined by

$$\chi(X) = \inf \{r>0 : \exists\ Y\subset E \text{ finite, } X\subset Y+rB_E\},$$

$$\beta(X) = \inf \{\varepsilon>0 : \exists\ Y\subset E \text{ weakly relatively compact, } X\subset Y+\varepsilon B_E\},$$

are the Hausdorff measure of noncompactness and the De Blasi measure of weak noncompactness [10], respectively.

Assume that $(E_i,\|\cdot\|_i)$ is a sequence of Banach spaces. Denote by $\ell^p(E_i)$, $1\le p<\infty$, the space of all sequences $x=(x_i)$, $x_i\in E_i$ for $i\in\mathbb{N}$, such thatc $\sum_{i=1}^{\infty} \|x_i\|_i^p < \infty$. Similarly, let $c_0=c_0(E_i)$ denote the space of all sequences $x=(x_i)$, $x_i\in E_i$, with the property $\|x_i\|_i \longrightarrow 0$ as $i \longrightarrow \infty$. Denote by π_n the projection operator

$$\pi_n\colon \ell^p(E_i) \longrightarrow E_n , \qquad \text{or } \pi_n\colon c_0(E_i) \longrightarrow E_n ,$$

defined by $\pi_n(x) = \pi_n(x_1, x_2, \dots) = x_n$.

Denote by τ_n the operator

$$\tau_n\colon \ell^p(E_i) \longrightarrow \ell^p(E_i) , \qquad \text{or } \tau_n\colon c_0(E_i) \longrightarrow c_0(E_i) ,$$

defined by $\tau_n(x) = \tau_n(x_1,x_2,\dots) = (o,\dots,o,x_n,x_{n+1},\dots)$.

Assume that χ_i (β_i) is the Hausdorff (De Blasi) measure in the space E_i. Let χ_p (β_p) denote the Hausdorff (De Blasi) measure in $\ell^p(E_i)$ and χ_o (β_o) denote the Hausdorff (De Blasi) measure in $c_o(E_i)$.

3. MAIN RESULTS. We consider the quantities

$$a(X) = \sup_{i\in\mathbb{N}} \chi_i(\pi_i X) , \qquad b(X) = \inf_{i\in\mathbb{N}} \|\tau_i X\| ,$$

$$c(X) = \sup_{i\in\mathbb{N}} \beta_i(\pi_i X) , \qquad d(X) = \inf_{i\in\mathbb{N}} \beta_o(\tau_n X) ,$$

where $X\in\mathcal{M}_{\ell^p(E_i)}$ (or $X\in\mathcal{M}_{c_0(E_i)}$), and we define

$$\mu_p(X) = \max \{a(X),b(X)\} , \qquad \gamma_p(X) = c(X) .$$

THEOREM . If $1 < p < \infty$, then

(1) μ_p is a measure of noncompactness in the space $\ell^p(E_i)$

(2) $\mu_p \leq \chi_p$

(3) μ_p is not equivalent to χ_p: does not exist a constant $c>0$ such that $c\beta_p \leq \gamma_p$, provided the spaces E_i are infinite dimensional

(4) $\chi_0 = max\ \{a(X),b(X)\}$

THEOREM . If $1 < p < \infty$, then

(1) γ_p is a measure of weak noncompactness in the space $\ell^p(E_i)$,

(2) $\gamma_p \leq \beta_p$

(3) γ_p is not equivalent to β_p: does not exist a constant $c>0$ such that $c\beta_p \leq \gamma_p$, provided the spaces E_i are nonreflexive and have the Schur property

(4) $\beta_o = max\ \{c(X),d(X)\}$

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